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# Monomial orthogonal polynomials of several variables☆

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**Abstract**

A monomial orthogonal polynomial of several variables is of the form $x^\alpha - Q_\alpha(x)$ for a multiindex $\alpha \in \mathbb{N}_0^{d+1}$ and it has the least $L^2$ norm among all polynomials of the form $x^\alpha - P(x)$, where $P$ and $Q_\alpha$ are polynomials of degree less than the total degree of $x^\alpha$. We study monomial orthogonal polynomials with respect to the weight function $\prod_{i=1}^{d+1} |x_i|^{2\kappa_i}$ on the unit sphere $S^d$ as well as for the related weight functions on the unit ball and on the standard simplex. The results include explicit formula, $L^2$ norm, and explicit expansion in terms of known orthonormal basis. Furthermore, in the case of $\kappa_1 = \cdots = \kappa_{d+1}$, an explicit basis for symmetric orthogonal polynomials is also given.


*MSC:* 33C50; 42C10

*Keywords:* $h$-harmonics; Orthogonal polynomials of several variables; Best $L^2$ approximation; Symmetric orthogonal polynomials

## 1. Introduction

The purpose of this paper is to study monomial orthogonal polynomials of several variables. Let $W$ be a weight function defined on a set $\Omega$ in $\mathbb{R}^d$. Let $\alpha \in \mathbb{N}_0^d$. The monomial orthogonal polynomials are of the form $R_\alpha(x) = x^\alpha - Q_\alpha(x)$ with $Q_\alpha$ being a polynomial

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of degree less than $n = |\alpha| := \alpha_1 + \cdots + \alpha_d$, and it is orthogonal to all polynomials of degree less than $n$ in $L^2(W, \Omega)$; in other words, they are orthogonal projections of $x^\alpha$ onto the subspace of orthogonal polynomials of degree $n$. In the case of one variable, such a polynomial is just an orthogonal polynomial normalized with a unit leading coefficient and its explicit formula is known for many classical weight functions. For several variables, there are many linearly independent orthogonal polynomials of the same degree and the explicit formula of $R_\alpha$ is not immediately known.

Let $\Pi_n^d$ denote the space of polynomials of degree at most $n$ in $d$ variables. The polynomial $R_\alpha$ can be considered as the error of the best approximation of $x^\alpha$ by polynomials from $\Pi_{n-1}^d$, $n = |\alpha|$, in $L^2(W, \Omega)$. Indeed, a standard Hilbert space argument shows that

$$\|R_\alpha\|_2 = \|x_\alpha - Q_\alpha\|_2 = \inf_P \{\|x^\alpha - P\|_2,\ P \in \Pi_{n-1}^d,\ n = |\alpha|\},$$

where $\|\cdot\|_2$ is the $L^2(W, \Omega)$ norm. In other words, $R_\alpha$ has the least $L^2$ norm among all polynomials of the form $x^\alpha - P$, where $P \in \Pi_{n-1}^d$.

Let $d\omega$ be the surface measure on the unit sphere $S^d = \{x : \|x\| = 1\}$, where $\|x\|$ is the Euclidean norm of $x \in \mathbb{R}^{d+1}$. In the present paper, we consider the monomial orthogonal polynomials in $L^2(h_\kappa^2\, d\omega, S^d)$, where

$$h_\kappa(x) = \prod_{i=1}^{d+1} |x_i|^{\kappa_i}, \quad \kappa_i \geqslant 0, \quad x \in \mathbb{R}^{d+1}. \tag{1.1}$$

The homogeneous orthogonal polynomials with respect to this weight function are called $h$-harmonics; they are the simplest examples of the $h$-harmonics associated with the reflection groups (see, for example, [4,5,7] and references therein). The weight function in (1.1) is invariant under the group $\mathbb{Z}_2^{d+1}$. Let $\mathcal{P}_n^{d+1}$ denote the space of homogeneous polynomials of degree $n$ in $d+1$ variables. The monomial homogeneous polynomials are of the form $R_\alpha(x) = x^\alpha - \|x\|^2 Q_\alpha(x)$, where $Q_\alpha \in \mathcal{P}_{n-2}^{d+1}$ and $n = |\alpha|$. In this case, we define $R_\alpha$ through a generating function and derive their various properties. Using a correspondence between the $h$-harmonics and orthogonal polynomials on the unit ball $B^d = \{x : \|x\| \leqslant 1\}$ of $\mathbb{R}^d$, this also gives the monomial orthogonal polynomials with respect to the weight function

$$W_\kappa^B(x) = \prod_{i=1}^{d} |x_i|^{2\kappa_i} (1 - \|x\|^2)^{\kappa_{d+1} - 1/2}, \quad x \in B^d, \quad \kappa_i \geqslant 0. \tag{1.2}$$

In the case $\kappa_i = 0$ for $1 \leqslant i \leqslant d$ and $\kappa_{d+1} = \mu$, the weight function $W_\kappa^B$ is the classical weight function $(1 - \|x\|^2)^{\mu - 1/2}$ for which the monomial polynomials are known already to Hermite (in special cases); see [8, vol. 2, Chapter 12]. There is also a correspondence between the $h$-harmonics and orthogonal polynomials on the simplex $T^d = \{x : x_i \geqslant 0, 1 - |x| \geqslant 0\}$ of $\mathbb{R}^d$, where $|x| = x_1 + \cdots + x_d$, which allows us to derive properties of the monomial orthogonal polynomials with respect to the weight function

$$W_\kappa^T(x) = \prod_{i=1}^{d} |x_i|^{\kappa_i - 1/2} (1 - |x|)^{\kappa_{d+1} - 1/2}, \quad x \in T^d, \quad \kappa_i \geqslant 0. \tag{1.3}$$

For these families of the weight functions, we will define the monomial orthogonal polynomials using generating functions, and give explicit formulae for these polynomials in the next section.

If $\kappa_1 = \cdots = \kappa_{d+1}$, then the weight function is invariant under the action of the symmetric group. We can consider the subspace of $h$-harmonics invariant under the symmetric group. Recently, in [6], Dunkl gave an explicit basis in terms of monomial symmetric polynomials. Another explicit basis can be derived from the explicit formulae of $R_\alpha$, which we give in Section 3.

Various explicit bases of orthogonal polynomials for the above weight functions have appeared in [7,11,16], some can be traced back to Refs. [2,8] in special cases. Our emphasis is on the monomial bases and explicit computation of the $L^2$ norm. The $L^2$ norms of the monomial orthogonal polynomials give the error of the best approximation to monomials by polynomials of lower degrees. We compute the norms in Section 4. They are expressed as integrals of the product of the Jacobi or Gegenbauer polynomials. We mention two special cases of our general results, in which $P_n(t)$ denotes the Legendre polynomial of degree $n$:

**Theorem 1.1.** *For $\alpha \in \mathbb{N}_0^d$, let $n = |\alpha|$. Then*

$$\min_{Q \in \Pi_{n-1}^d} \frac{1}{\operatorname{vol} B^d} \int_{B^d} |x^\alpha - Q(x)|^2 \, dx = \frac{d\alpha!}{2^n (d/2)_n} \int_0^1 \prod_{i=1}^d P_{\alpha_i}(t) t^{n+d-1} \, dt,$$

*where* $\operatorname{vol} B^d = \pi^{d/2}/\Gamma(d/2+1)$ *is the volume of $B^d$, and*

$$\min_{Q \in \Pi_{n-1}^d} \frac{1}{d!} \int_{T^d} |x^\alpha - Q(x)|^2 \, dx = \frac{d\alpha!^2}{(d)_{2n}} \int_0^1 \prod_{i=1}^d P_{\alpha_i}(2r-1) r^{n+d-1} \, dr.$$

As the best approximation to $x^\alpha$, the monomial orthogonal polynomials with respect to the unit weight function (Lebesgue measure) on $B^2$ have been studied recently in [3]. Let us also mention [1], in which certain invariant polynomials with the least $L^p$ norm on $S^d$ are studied.

For $h$-harmonics, the set $\{R_\alpha : |\alpha| = n\}$ contains an orthogonal basis of $h$-harmonics of degree $n$ but the set itself is not a basis. In general, two monomial orthogonal polynomials of the same degree are not orthogonal to each other. On the other hand, for each of the three families of the weight functions, an orthonormal basis can be given in terms of the Jacobi polynomials or the Gegenbauer polynomials. We will derive an explicit expansion of $R_\alpha$ in terms of this orthonormal basis in Section 5, the coefficients of the expansion are given in terms of Hahn polynomials of several variables.

Finally in Section 6, we discuss another property of the polynomials defined by the generating function. It leads to an expansion of monomials in terms of monomial orthogonal polynomials.

## 2. Monomial orthogonal polynomials

Throughout this paper we use the standard multiindex notation. For $\alpha \in \mathbb{N}_0^m$ we write $|\alpha| = \alpha_1 + \cdots + \alpha_m$. For $\alpha, \beta \in \mathbb{N}_0^m$ we also write $\alpha! = \alpha_1! \cdots \alpha_m!$ and $(\alpha)_\beta =$

$(\alpha_1)_{\beta_1}\cdots(\alpha_m)_{\beta_m}$, where $(a)_n = a(a+1)\dots(a+n-1)$ is the Pochhammer symbol. Furthermore, for $\alpha \in \mathbb{N}^m$ and $a, b \in \mathbb{R}$, we write $a\alpha + b\mathbf{1} = (a\alpha_1 + b, \dots, a\alpha_m + b)$ and denote $\mathbf{1} := (1, 1, \dots, 1)$. For $\alpha, \beta \in \mathbb{N}_0^m$, the inequality $\alpha \leqslant \beta$ means that $\alpha_i \leqslant \beta_i$ for $1 \leqslant i \leqslant m$.

### 2.1. Monomial *h*-harmonics

First, we recall relevant part of the theory of $h$-harmonics; see [4,5,7] and the reference therein. We shall restrict ourself to the case of $h_\kappa$ defined in (1.1); see also [16].

Let $\mathcal{H}_n^{d+1}(h_\kappa^2)$ denote the space of homogeneous orthogonal polynomials of degree $n$ with respect to $h_\kappa^2\, d\omega$ on $S^d$. If all $\kappa_i = 0$, then $\mathcal{H}_n^{d+1}(h_\kappa^2)$ is just the space of the ordinary harmonics. It is known that

$$\dim\, \mathcal{H}_n^{d+1}(h_\kappa^2) = \dim\, \mathcal{P}_n^{d+1} - \dim\, \mathcal{P}_{n-2}^{d+1} = \binom{n+d}{d} - \binom{n+d-2}{d}.$$

The elements of $\mathcal{H}_n^{d+1}(h_\kappa^2)$ are called $h$-harmonics since they can be defined through an analog of Laplacian operator. The essential ingredient is Dunkl's operators, which are a family of first-order differential-difference operators defined by

$$\mathcal{D}_i f(x) = \partial_i f(x) + \kappa_i \frac{f(x) - f(x_1, \dots, -x_i, \dots, x_{d+1})}{x_i}, \quad 1 \leqslant i \leqslant d+1. \tag{2.1}$$

These operators commute; that is, $\mathcal{D}_i\mathcal{D}_j = \mathcal{D}_j\mathcal{D}_i$. The $h$-Laplacian is defined by $\Delta_h = \mathcal{D}_1^2 + \cdots + \mathcal{D}_{d+1}^2$. Then $\Delta_h P = 0$, $P \in \mathcal{P}_n^{d+1}$ if and only if $P \in \mathcal{H}_n^{d+1}(h_\kappa^2)$. The structure of the $h$-harmonics and that of ordinary harmonic polynomials are parallel. Some of the properties of $h$-harmonics can be expressed using the *intertwining operator*, $V_\kappa$, which is a linear operator that acts between ordinary harmonics and $h$-harmonics. It is uniquely determined by the properties

$$\mathcal{D}_i V_\kappa = V_\kappa \partial_i, \quad V_\kappa 1 = 1, \quad V_\kappa \mathcal{P}_n^{d+1} \subset \mathcal{P}_n^{d+1}.$$

For the weight function $h_\kappa$ in (1.1), $V_\kappa$ is an integral operator defined by

$$\begin{aligned} V_\kappa f(x) = &\int_{[-1,1]^{d+1}} f(x_1 t_1, \dots, x_{d+1} t_{d+1}) \\ &\times \prod_{i=1}^{d+1} c_{\kappa_i} (1+t_i)(1-t_i^2)^{\kappa_i - 1}\, dt, \end{aligned} \tag{2.2}$$

where $c_\mu = \Gamma(\mu + 1/2)/(\sqrt{\pi}\,\Gamma(\mu))$. If any one of $\kappa_i = 0$, the formula holds under the limit

$$\lim_{\mu \to 0} c_\mu \int_{-1}^{1} f(t)(1-t^2)^{\mu-1}\, d\mu = [f(1) + f(-1)]/2. \tag{2.3}$$

The Poisson kernel, or reproducing kernel, $P(h_\kappa^2; x, y)$ of the $h$-harmonics is defined by the property

$$f(x) = c_h' \int_{S^d} f(y) P(h_\kappa^2; x, y) f(y) h_\kappa^2(y)\, d\omega(y),$$
$$c_h' = \frac{\Gamma(|\kappa| + \frac{d+1}{2})}{2 \prod_{i=1}^{d+1} \Gamma(\kappa_i + \frac{1}{2})} \tag{2.4}$$

for $f \in \mathcal{H}_n^d(h_\kappa^2)$ and $\|y\| \leqslant 1$, where $c_h'$ is the normalization constant of the weight function $h_\kappa^2$ on the unit sphere $S^d$, $c_h' \int_{S^d} h_\kappa^2\, d\omega = 1$ and $d\omega$ is the surface measure. Using the intertwining operator, the Poisson kernel of the $h$-harmonics can be written as

$$P(h_\kappa^2; x, y) = V_\kappa \left[ \frac{1 - \|y\|^2}{(1 - 2\langle y, \cdot \rangle + \|y\|^2)^{\rho+1}} \right](x), \quad \rho = |\kappa| + \frac{d-1}{2}$$

for $\|y\| < 1 = \|x\|$. If all $\kappa_i = 0$, then $V_\kappa = id$ is the identity operator and $P(h_0^2; x, y)$ is the classical Poisson kernel, which is related to the Poisson kernel of the Gegenbauer polynomials

$$\frac{1 - r^2}{(1 - 2rt + r^2)^{\lambda+1}} = \sum_{n=0}^\infty \frac{n + \lambda}{\lambda} C_n^\lambda(t) r^n.$$

The above function can be viewed as a generating function for the Gegenbauer polynomials $C_n^\lambda(t)$. The usual generating function of $C_n^\lambda$, however, takes the following form: $(1 - 2rt + r^2)^{-\lambda} = \sum_{n=0}^\infty C_n^\lambda(t) r^n$. Our definition of the monomial orthogonal polynomials is the analog of the generating function of $C_n^\lambda$ in several variables.

**Definition 2.1.** Let $\rho = |\kappa| + \frac{d-1}{2} > 0$. Define polynomials $\widetilde{R}_\alpha(x)$ by

$$V_\kappa \left[ \frac{1}{(1 - 2\langle b, \cdot \rangle + \|b\|^2 \|x\|^2)^\rho} \right](x) = \sum_{\alpha \in \mathbb{N}_0^{d+1}} b^\alpha \widetilde{R}_\alpha(x), \quad x \in \mathbb{R}^{d+1}.$$

Let $F_B$ be the Lauricella hypergeometric series of type $B$, which generalizes the hypergeometric function ${}_2F_1$ to several variables (cf. [10]),

$$F_B(\alpha, \beta; c; x) = \sum_\gamma \frac{(\alpha)_\gamma (\beta)_\gamma}{(c)_{|\gamma|} \gamma!} x^\gamma, \quad \alpha, \beta \in \mathbb{N}_0^{d+1}, \ c \in \mathbb{R}, \quad \max_{1 \leqslant i \leqslant d+1} |x_i| < 1,$$

where the summation is taken over $\gamma \in \mathbb{N}_0^{d+1}$. We derive properties of $\widetilde{R}_\alpha$ in the following.

**Proposition 2.2.** *The polynomials $\widetilde{R}_\alpha$ satisfy the following properties*:

(1) $\widetilde{R}_\alpha \in \mathcal{P}_n^{d+1}$ *and*

$$\widetilde{R}_\alpha(x) = \frac{2^{|\alpha|} (\rho)_{|\alpha|}}{\alpha!} \sum_\gamma \frac{(-\alpha/2)_\gamma (-(\alpha + \mathbf{1})/2)_\gamma}{(-|\alpha| - \rho + 1)_{|\gamma|} \gamma!} \|x\|^{2|\gamma|} V_\kappa(x^{\alpha - 2\gamma}),$$

*where the series terminates as the summation is over all $\gamma$ such that $2\gamma \leqslant \alpha$;*

(2) $\widetilde{R}_\alpha \in \mathcal{H}_n^{d+1}(h_\kappa^2)$ *and* $\widetilde{R}_\alpha(x) = \dfrac{2^{|\alpha|}(\rho)_{|\alpha|}}{\alpha!} V_\kappa[S_\alpha(\cdot)](x)$ *for* $\|x\| = 1$, *where*

$$S_\alpha(y) = y^\alpha F_B\left(-\frac{\alpha}{2}, \frac{-\alpha+\mathbf{1}}{2}; -|\alpha| - \rho + 1; \frac{1}{y_1^2}, \dots, \frac{1}{y_{d+1}^2}\right).$$

*Furthermore,*

$$\sum_{|\alpha|=n} b^\alpha \widetilde{R}_\alpha(x) = \frac{\rho}{n+\rho} P_n(h_\kappa^2; b, x), \quad \|x\| = 1,$$

*where* $P_n(h_\kappa^2; y, x)$ *is the reproducing kernel of* $\mathcal{H}_n^{d+1}(h_\kappa^2)$ *in* $L^2(h_\kappa^2, S^{d-1})$.

**Proof.** Using the multinomial and binomial formula, we write

$$\begin{aligned}(1 - 2\langle a, y\rangle + \|a\|^2)^{-\rho} &= (1 - a_1(2y_1 - a_1) - \cdots - a_d(2y_{d+1} - a_{d+1}))^{-\rho} \\ &= \sum_\beta \frac{(\rho)_{|\beta|}}{\beta!} a^\beta (2y_1 - a_1)^{\beta_1} \dots (2y_{d+1} - a_{d+1})^{\beta_{d+1}} \\ &= \sum_\beta \frac{(\rho)_{|\beta|}}{\beta!} \sum_\gamma \frac{(-\beta_1)_{\gamma_1} \dots (-\beta_{d+1})_{\gamma_{d+1}}}{\gamma!} \\ &\quad \times 2^{|\beta|-|\gamma|} y^{\beta-\gamma} a^{\gamma+\beta}.\end{aligned}$$

Changing summation indices $\beta_i + \gamma_i = \alpha_i$ and using the formulae

$$(\rho)_{m-k} = \frac{(-1)^k (\rho)_m}{(1-\rho-m)_k} \quad \text{and} \quad \frac{(-m+k)_k}{(m-k)!} = \frac{(-1)^k(-m)_{2k}}{m!}$$

as well as $2^{-2k}(-m)_{2k} = (-m/2)_k((1-m)/2)_k$, we can rewrite the formula as

$$\begin{aligned}(1 - 2\langle a, y\rangle + \|a\|^2)^{-\rho} &= \sum_\alpha a^\alpha \frac{2^{|\alpha|}(\rho)_{|\alpha|}}{\alpha!} \sum_\gamma \frac{(-\alpha/2)_\gamma((-\alpha+1)/2)_\gamma}{(-|\alpha|-\rho+1)_{|\gamma|}\gamma!} y^{\alpha-2\gamma} \\ &= \sum a^\alpha \frac{2^{|\alpha|}(\rho)_{|\alpha|}}{\alpha!} y^\alpha F_B \\ &\quad \times \left(-\frac{\alpha}{2}, \frac{\mathbf{1}-\alpha}{2}; -|\alpha| - \rho + 1; \frac{1}{y_1^2}, \dots, \frac{\mathbf{1}}{y_d^2}\right).\end{aligned}$$

Using the first equal sign of the expansion with the function

$$(1 - 2\langle b, y\rangle + \|x\|^2\|b\|^2)^{-\rho} = (1 - 2\langle \|x\|b, y/\|x\|\rangle + \|\|x\|b\|^2)^{-\rho}$$

and applying $V$ with respect to $y$ gives the expression of $\widetilde{R}_\alpha$ in (1). If $\|x\| = 1$, then the second equal sign gives the expression of $\widetilde{R}_\alpha$ in (2). We still need to show that $\widetilde{R}_\alpha \in \mathcal{H}_n^{d+1}(h_\kappa^2)$. Let $\|x\| = 1$. For $\|y\| \leqslant 1$ the generating function of the Gegenbauer polynomials gives

$$\begin{aligned}(1 - 2\langle b, y\rangle + \|b\|^2)^{-\rho} &= (1 - 2\|b\|\langle b/\|b\|, y\rangle + \|b\|^2)^{-\rho} \\ &= \sum_{n=0}^\infty \|b\|^n C_n^\rho(\langle b/\|b\|, y\rangle).\end{aligned}$$

Consequently, applying $V_\kappa$ on $y$ in the above equation gives

$$\sum_{|\alpha|=n} b^\alpha \widetilde{R}_\alpha(x) = \|b\|^n V_\kappa[C_n^\rho(\langle b/\|b\|, \cdot\rangle)](x), \quad \|x\| = 1.$$

On the other hand, it is known that the reproducing kernel $P_n(h_\kappa^2; x, y)$ of $\mathcal{H}_n^{d+1}(h_\kappa^2)$ is given by [7, p. 190]

$$P_n(h_\kappa^2; x, y) = \frac{n+\rho}{\rho} \|y\|^n V_\kappa[C_n^\rho(\langle y/\|y\|, \cdot\rangle)](x), \quad \|y\| \leqslant \|x\| = 1,$$

so that $\sum_{|\alpha|=n} b^\alpha \widetilde{R}_\alpha(x)$ is a constant multiple of $P_n(h_\kappa^2; x, b)$. Consequently, for any $b$, $\sum b^\alpha \widetilde{R}_\alpha(x)$ is an element in $\mathcal{H}_n^{d+1}(h_\kappa^2)$; therefore, so is $\widetilde{R}_\alpha$. □

In the following let $[x]$ denote the integer part of $x$. We also use $[\alpha/2]$ to denote $([\alpha_1/2], \ldots, [\alpha_{d+1}/2])$ for $\alpha \in \mathbb{N}_0^{d+1}$.

**Proposition 2.3.** *Let $\rho = |\kappa| + (d-1)/2$. Then*

$$\widetilde{R}_\alpha(x) = \frac{2^{|\alpha|}(\rho)_{|\alpha|}}{\alpha!} \frac{(\mathbf{1}/2)_{\alpha-\beta}}{(\kappa+\mathbf{1}/2)_{\alpha-\beta}} R_\alpha(x), \quad \textit{where } \beta = \alpha - \left[\frac{\alpha+\mathbf{1}}{2}\right]$$

*and*

$$R_\alpha(x) = x^\alpha F_B\left(-\beta, -\alpha+\beta-\kappa+\frac{\mathbf{1}}{2}; -|\alpha|-\rho+1; \frac{\|x\|^2}{x_1^2}, \ldots, \frac{\|x\|^2}{x_{d+1}^2}\right).$$

**Proof.** By considering $m$ being even or odd, it is easy to verify that

$$c_\kappa \int_{-1}^{1} t^{m-2k}(1+t)(1-t^2)^{\kappa-1}\, dt = \frac{\left(\frac{1}{2}\right)_{[\frac{m+1}{2}]}}{\left(\kappa+\frac{1}{2}\right)_{[\frac{m+1}{2}]}} \frac{\left(-[\frac{m+1}{2}]-\kappa+\frac{1}{2}\right)_k}{\left(-[\frac{m+1}{2}]+\frac{1}{2}\right)_k}$$

for $\kappa \geqslant 0$. Hence, using the explicit formula of $V_\kappa$, the formula of $\widetilde{R}_\alpha$ in (1) of Proposition 2.2 becomes,

$$\begin{aligned}\widetilde{R}_\alpha(x) = {} & \frac{2^{|\alpha|}(\rho)_{|\alpha|}}{\alpha!} \frac{(\mathbf{1}/2)_{[\frac{\alpha+\mathbf{1}}{2}]}}{(\kappa+\mathbf{1}/2)_{[\frac{\alpha+\mathbf{1}}{2}]}} \\ & \times \sum_\gamma \frac{(-\alpha/2)_\gamma((-\alpha+\mathbf{1})/2)_\gamma}{(-|\alpha|-\rho+1)_{|\gamma|}\gamma!} \frac{(-[(\alpha+\mathbf{1})/2]-\kappa+\mathbf{1}/2)_\gamma}{(-[(\alpha+\mathbf{1})/2]+\mathbf{1}/2)_\gamma} \|x\|^{2|\gamma|} x^{\alpha-2\gamma}.\end{aligned}$$

Using the fact that

$$\left(-\frac{\alpha}{2}\right)_\gamma \left(\frac{-\alpha+\mathbf{1}}{2}\right)_\gamma = \left(-\alpha+\left[\frac{\alpha+\mathbf{1}}{2}\right]\right)_\gamma \left(-\left[\frac{\alpha+\mathbf{1}}{2}\right]+\frac{\mathbf{1}}{2}\right)_\gamma,$$

the above expression of $\widetilde{R}_\alpha$ can be written in terms of $F_B$ as stated. □

Note that the $F_B$ function in the proposition is a finite series, since $(-n)_m = 0$ if $m > n$. In particular, this shows that $R_\alpha(x)$ is the monomial orthogonal polynomial of the form $R_\alpha(x) = x^\alpha - \|x\|^2 Q_\alpha(x)$, where $Q_\alpha \in \mathcal{P}^d_{n-2}$.

Another generalization of the hypergeometric series ${}_2F_1$ to several variables is the Lauricella function of type $A$, defined by (cf. [10])

$$F_A(c, \alpha; \beta; x) = \sum_{\gamma} \frac{(c)_{|\gamma|}(\alpha)_\gamma}{(\beta)_\gamma \gamma!} x^\gamma, \quad \alpha, \beta \in \mathbb{N}_0^{d+1}, \ c \in \mathbb{R},$$

where the summation is taken over $\gamma \in \mathbb{N}_0^{d+1}$. If all components of $\alpha$ are even, then we can write $R_\alpha$ using $F_A$.

**Proposition 2.4.** *Let $\beta \in \mathbb{N}_0^{d+1}$. Then*

$$R_{2\beta}(x) = (-1)^{|\beta|} \frac{(\kappa + \mathbf{1}/2)_\beta}{(n+\rho)_{|\beta|}} \|x\|^{|2\beta|} F_A\left(-\beta, |\beta| + \rho; \kappa + \frac{\mathbf{1}}{2}; \frac{x_1^2}{\|x\|^2}, \ldots, \frac{x_{d+1}^2}{\|x\|^2}\right).$$

**Proof.** For $\alpha = 2\beta$ the formula in terms of $F_B$ becomes

$$R_{2\beta}(x) = \sum_{\gamma \leqslant \beta} \frac{(-\beta)_\gamma(-\beta - \kappa + \mathbf{1}/2)_\gamma}{(-2|\beta| - \rho + 1)_{|\gamma|}\gamma!} \|x\|^{2|\gamma|} x^{2\beta - 2\gamma},$$

where $\gamma \leqslant \beta$ means $\gamma_1 < \beta_1, \ldots, \gamma_{d+1} < \beta_{d+1}$; note that $(-\beta)_\gamma = 0$ if $\gamma > \beta$. Changing the summation index by $\gamma_i \mapsto \beta_i - \gamma_i$ and using the formula $(a)_{n-m} = (-1)^m (a)_n/(1-n-a)_m$ to rewrite the Pochhammer symbols, for example,

$$(\kappa + \mathbf{1}/2)_{\beta - \gamma} = \frac{(-1)^\gamma (\kappa + \mathbf{1}/2)_\beta}{(-\beta - \kappa + \mathbf{1}/2)_\gamma}, \quad (\beta - \gamma)! = (\mathbf{1})_{\beta-\gamma} = \frac{(-1)^{|\gamma|}\beta!}{(-\beta)_\gamma},$$

we can rewrite the summation into the stated formula in $F_A$. □

Let $\mathrm{proj}_n : \mathcal{P}_n^{d+1} \mapsto \mathcal{H}_n^{d+1}(h_\kappa^2)$ denote the projection operator of polynomials in $\mathcal{P}_n^{d+1}$ onto $\mathcal{H}_n^{d+1}(h_\kappa^2)$. It follows that $R_\alpha$ is the orthogonal projection of the monomial $x^\alpha$. Recall that $\mathcal{D}_i$ is the Dunkl operator defined in (2.1). We define $\mathcal{D}^\alpha = \mathcal{D}_1^{\alpha_1} \cdots \mathcal{D}_{d+1}^{\alpha_{d+1}}$ for $\alpha \in \mathbb{N}_0^{d+1}$. Let $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ denote the standard basis of $\mathbb{R}^{d+1}$.

**Proposition 2.5.** *The polynomials $R_\alpha$ satisfy the following properties*:

(1) $R_\alpha(x) = \mathrm{proj}_n\, x^\alpha$, $n = |\alpha|$, *and*

$$R_\alpha(x) = \frac{(-1)^n}{2^n(\rho)_n} \|x\|^{2\rho + 2n} \mathcal{D}^\alpha \left(\|x\|^{-2\rho}\right), \quad \rho = |\kappa| + \frac{d-1}{2}.$$

(2) $R_\alpha$ *satisfies the relation*

$$\|x\|^2 \mathcal{D}_i R_\alpha(x) = -2(n+\rho)\left[R_{\alpha + e_i}(x) - x_i R_\alpha(x)\right].$$

(3) *The set* $\{R_\alpha : |\alpha| = n, \alpha_{d+1} = 0, 1\}$ *is a basis of* $\mathcal{H}_n^{d+1}(h_\kappa^2)$.

**Proof.** Since $R_\alpha \in \mathcal{H}_n^{d+1}(h_\kappa^2)$ and $R_\alpha(x) = x^\alpha - \|x\|^2 Q(x)$, where $Q \in \mathcal{P}_{n-2}^{d+1}$, it follows that $R_\alpha(x) = \operatorname{proj}_n x^\alpha$. On the other hand, it is shown in [19] that the polynomials $H_\alpha$, defined by

$$H_\alpha(x) = \|x\|^{2\rho+2n} \mathcal{D}^\alpha \|x\|^{-2\rho},$$

satisfy the relation $H_\alpha(x) = (-1)^n 2^n (\rho)_n \operatorname{proj}_{|\alpha|} x^\alpha$, from which the explicit formula in (1) follows. The polynomials $H_\alpha$ satisfy the recursive relation

$$H_{\alpha+e_i}(x) = -(2|\alpha| + 2\rho) x_i H_\alpha(x) + \|x\|^2 \mathcal{D}_i H_\alpha(x),$$

which gives the relation in (2). Finally, it is proved in [19] that $\{H_\alpha : |\alpha| = n, \alpha_{d+1} = 0, 1\}$ is a basis of $\mathcal{H}_n^{d+1}(h_\kappa^2)$. □

In the case of $\alpha = ne_i$, $R_\alpha$ takes a simple form. Indeed, let $C_n^{(\lambda,\mu)}(t)$ denote the generalized Gegenbauer polynomials defined by

$$C_n^{(\lambda,\mu)}(x) = c_\mu \int_{-1}^{1} C_n^\lambda(xt)(1+t)(1-t^2)^{\mu-1}\,dt.$$

These polynomials are orthogonal with respect to the weight function $w_{\lambda,\mu}(t) = |t|^{2\mu}(1-t^2)^{\lambda-1/2}$ on $[-1, 1]$ and they become Gegenbauer polynomials when $\mu = 0$ (use (2.3)); that is, $C_n^{(\lambda,0)}(t) = C_n^\lambda(t)$. In terms of the Jacobi polynomials $P_n^{(a,b)}(t)$, the generalized Gegenbauer polynomials can be written as

$$\begin{aligned} C_{2n}^{(\lambda,\mu)}(x) &= \frac{(\lambda+\mu)_n}{\left(\mu+\frac{1}{2}\right)_n} P_n^{(\lambda-1/2,\mu-1/2)}(2x^2-1), \\ C_{2n+1}^{(\lambda,\mu)}(x) &= \frac{(\lambda+\mu)_{n+1}}{\left(\mu+\frac{1}{2}\right)_{n+1}} x P_n^{(\lambda-1/2,\mu+1/2)}(2x^2-1). \end{aligned} \tag{2.5}$$

Recall that the Jacobi polynomial $P_n^{(a,b)}(t)$ can be written as a ${}_2F_1$ function

$$P_n^{(a,b)}(t) = \frac{(a+1)_n}{n!} {}_2F_1\left(\begin{matrix} -n, n+a+b+1 \\ a+1 \end{matrix}; \frac{1-t}{2}\right). \tag{2.6}$$

For $\alpha = ne_i$, the $F_B$ formula of $R_\alpha$ becomes a single sum since $(-m)_j = 0$ if $m < j$, which can be written in terms of ${}_2F_1$. For example, if $n = 2m+1$, then

$$\begin{aligned} R_{(2m+1)e_1}(x) &= \sum_{j=0}^{m} \frac{(-m)_j(-m-\kappa_1-1/2)_j}{(-2m-1-\rho+1)_j\, j!} \|x\|^{2j} x_1^{2m-2j+1} \\ &= x_1^{2m+1} {}_2F_1\left(\begin{matrix} -m, -m-\kappa_1-1/2 \\ -2m-1-\rho+1 \end{matrix}; \frac{\|x\|^2}{x_1^2}\right). \end{aligned}$$

This can be written in terms of Jacobi polynomials (2.6), upon changing the summation index by $j \mapsto m - j$, and further in the generalized Gegenbauer polynomials using (2.5). The result is

**Corollary 2.6.** *Let $n \in \mathbb{N}_0$. Then $R_{ne_i}(x) = \mathrm{proj}_n\, x_i^n$ satisfies*

$$R_{ne_i}(x) = \|x\|^n \left[k_n^{(\rho-\kappa_i,\kappa_i)}\right]^{-1} C_n^{(\rho-\kappa_i,\kappa_i)}(x_i\|x\|),$$

*where $k_n^{(\lambda,\mu)}$ denote the leading coefficient of $C_n^{(\lambda,\mu)}(t)$ given by*

$$k_{2n}^{(\lambda,\mu)} = \frac{(\lambda+\mu)_{2n}}{\left(\mu+\frac{1}{2}\right)_n n!} \quad \textit{and} \quad k_{2n+1}^{(\lambda,\mu)} = \frac{(\lambda+\mu)_{2n+1}}{\left(\mu+\frac{1}{2}\right)_{n+1} n!}. \tag{2.7}$$

In the case of ordinary harmonics, that is, $\kappa_i = 0$, the polynomials $R_{ne_i}$ are given in terms of the Gegenbauer polynomials.

### *2.2. Monomial orthogonal polynomials on the unit ball*

The $h$-spherical harmonics associated to (1.1) are closely related to orthogonal polynomials associated to the weight functions $W_\kappa^B$ in (1.2). In fact, if $Y \in \mathcal{H}_n^{d+1}(h_\kappa^2)$ is an $h$-harmonic associated with $h_\kappa(y) = \prod_{i=1}^{d+1} |y_i|^{\kappa_i}$ that is even in its $(d+1)$th variable, $Y(y', y_{d+1}) = Y(y', -y_{d+1})$, then the polynomial $P_\alpha$ defined by

$$Y(y) = r^n P(x), \quad y = r(x, x_{d+1}), \quad r = \|y\|,\ (x, x_{d+1}) \in S^d, \tag{2.8}$$

is an orthogonal polynomials with respect to $W_\kappa^B$. Moreover, this defines an one-to-one correspondence between the two sets of polynomials [17].

Working with polynomials on $B^d$, the monomials are $x^\alpha$ with $\alpha \in \mathbb{N}_0^d$, instead of $\mathbb{N}_0^{d+1}$. Since $x_{d+1}^2 = 1 - \|x\|^2$ for $(x, x_{d+1}) \in S^d$, we only consider $R_\alpha$ in Definition 2.1 with $\alpha = (\alpha_1, \ldots, \alpha_d, 0)$. The correspondence (2.8) leads to the following definition:

**Definition 2.7.** Let $\rho = |\kappa| + \frac{d-1}{2}$. Define polynomials $\widetilde{R}_\alpha^B(x)$, $\alpha \in \mathbb{N}_0^d$, by

$$c_\kappa \int_{[-1,1]^d} \frac{1}{(1 - 2(b_1x_1t_1 + \cdots + b_dx_dt_d) + \|b\|^2)^\rho} \prod_{i=1}^d (1+t_i)(1-t_i^2)^{\kappa_i - 1}\, dt$$
$$= \sum_{\alpha \in \mathbb{N}_0^d} b^\alpha \widetilde{R}_\alpha^B(x), \quad x \in B^d.$$

The polynomials $\widetilde{R}_\alpha^B$ form a basis of the subspace of orthogonal polynomials of degree $n$ with respect to $W_\kappa^B$. It is given by the explicit formula

**Proposition 2.8.** *Let $\rho = |\kappa| + (d-1)/2$. For $\alpha \in \mathbb{N}_0^d$ and $x \in \mathbb{R}^d$,*

$$\widetilde{R}_\alpha^B(x) = \frac{2^{|\alpha|}(\rho)_{|\alpha|}}{\alpha!} \frac{(\mathbf{1}/2)_{\alpha-\beta}}{(\kappa+\mathbf{1}/2)_{\alpha-\beta}} R_\alpha^B(x), \quad \textit{where } \beta = \alpha - \left[\frac{\alpha+\mathbf{1}}{2}\right]$$

*and*

$$R_\alpha(x) = x^\alpha F_B\left(-\beta, -\alpha+\beta-\kappa+\frac{\mathbf{1}}{2}; -|\alpha|-\rho+1; \frac{1}{x_1^2}, \ldots, \frac{1}{x_d^2}\right).$$

*In particular,* $R_\alpha^B(x) = x^\alpha - Q_\alpha(x)$, $Q_\alpha \in \Pi_{n-1}^d$, *is the monomial orthogonal polynomial with respect to* $W_\kappa^B$ *on* $B^d$.

**Proof.** Setting $b_{d+1} = 0$ and $\|x\| = 1$ in the generating function (2.1) shows that the generating function of $\widetilde{R}_\alpha^B$ is the same as the one for $\widetilde{R}_{(\beta,0)}(x)$. Consequently, $\widetilde{R}_\alpha^B(x) = \widetilde{R}_{(\alpha,0)}(x, x_{d+1})$ for $(x, x_{d+1}) \in S^d$. Since $\widetilde{R}_{(\alpha,0)}(x, x_{d+1})$ is even in its $d+1$ variable, the correspondence (2.8) shows that $\widetilde{R}_\alpha^B$ is orthogonal and its properties can be derived from those of $\widetilde{R}_\alpha$. □

In particular, if $\kappa_i = 0$ for $i = 1, \ldots, d$ and $\kappa_{d+1} = \mu$ so that $W_\kappa^B$ becomes the classical weight function $(1 - \|x\|^2)^{\mu-1/2}$, then the limit relation (2.3) shows that the generating function becomes simply

$$(1 - 2\langle b, x\rangle + \|b\|^2)^{-\mu-(d-1)/2} = \sum_{\alpha \in \mathbb{N}_0^d} b^\alpha R_\alpha^B(x), \quad x \in \mathbb{R}^d.$$

This is the generating function of one family of Appell's biorthogonal polynomials and $R_\alpha^B(x)$ is usually denoted by $V_\alpha(x)$ in the literature (see, for example, [8, vol. II, Chapter 12] or [7, Chapter 2]).

The definition of $R_\alpha^B$ comes from that of $h$-harmonics $R_{(\alpha,0)}$ by the correspondence. If we consider $R_\beta$ with $\beta = (\alpha, \alpha_{d+1})$ and assume that $\alpha_{d+1}$ is an even integer, then $R_\beta$ leads to the orthogonal projection of the polynomial $x^\alpha(1 - \|x\|^2)^{\alpha_{d+1}/2}$ with respect to $W_\kappa^B$ on $B^d$. Furthermore, the correspondence also gives a generating function of these projections.

### 2.3. *Monomial orthogonal polynomials on the simplex*

The $h$-spherical harmonics associated to (1.1) are also related to orthogonal polynomials associated to the weight functions $W_\kappa^T$ in (1.3). If $Y \in \mathcal{H}_{2n}^{d+1}(h_\kappa^2)$ is an $h$-harmonic that is even in each of its variables, then $Y$ can be written as

$$Y(y) = r^n P(x_1^2, \ldots, x_d^2), \quad y = r(x_1, \ldots, x_d, x_{d+1}), \quad r = \|y\|. \tag{2.9}$$

The polynomial $P(x)$, $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ is an orthogonal polynomial of degree $n$ in $d$ variables with respect to $W_\kappa^T$ on $T^d$. Moreover, this defines an one-to-one correspondence between the two sets of polynomials [18].

Since the simplex $T$ has a natural symmetry in terms of $(x_1, \ldots, x_d, x_{d+1})$, $x_{d+1} = 1 - |x|$, we use the homogeneous coordinates $X := (x_1, \ldots, x_d, x_{d+1})$. For the monomial $h$-harmonics defined in Definition 2.1, the polynomial $R_{2\alpha}$ is even in each of its variables, which corresponds to, under (2.9), monomial orthogonal polynomials $R_\alpha^T$ in $\mathcal{V}_n^d(W_\kappa^T)$ in the homogeneous coordinates $X$. This leads to the following definition:

**Definition 2.9.** Let $\rho = |\kappa| + \frac{d-1}{2}$. Define polynomials $\widetilde{R}_\alpha^T(x)$, $\alpha \in \mathbb{N}_0^d$, by

$$c_\kappa \int_{[-1,1]^{d+1}} \frac{1}{(1 - 2(b_1 x_1 t_1 + \cdots + b_{d+1} x_{d+1} t_{d+1}) + \|b\|^2)^\rho} \prod_{i=1}^{d+1} (1 - t_i^2)^{\kappa_i - 1}\, dt$$

$$= \sum_{\alpha \in \mathbb{N}_0^{d+1}} b^{2\alpha} \widetilde{R}_\alpha^T(x), \quad x \in T^d, \quad x_{d+1} = 1 - |x|.$$

The main properties of $R_\alpha^T$ are summarized in the following proposition.

**Proposition 2.10.** *For each $\alpha \in \mathbb{N}_0^{d+1}$ with $|\alpha| = n$, the polynomials*

$$\widetilde{R}_\alpha^T(x) = \frac{2^{2|\alpha|}(\rho)_{2|\alpha|}}{(2\alpha)!} \frac{(\mathbf{1}/2)_\alpha}{(\kappa + \mathbf{1}/2)_\alpha} R_\alpha^T(x),$$

*where*

$$R_\alpha^T(x) = X^\alpha F_B\left(-\alpha, -\alpha - \kappa + \frac{\mathbf{1}}{2}; -2|\alpha| - \rho + 1; \frac{1}{x_1}, \ldots, \frac{1}{x_{d+1}}\right)$$
$$= (-1)^n \frac{(\kappa + \mathbf{1})_\alpha}{(n + |\kappa| + d)_n} F_A(|\alpha| + |\kappa| + d, -\alpha; \kappa + \mathbf{1}; X)$$

*are orthogonal polynomials with respect to $W_\kappa^T$ on the simplex $T^d$. Moreover, $R_\alpha^T(x) = X^\alpha - Q_\alpha(x)$, where $Q_\alpha$ is a polynomial of degree at most $n-1$, and $\{R_\alpha^T, \alpha = (\alpha', 0), |\alpha| = n\}$ is a basis for the subspace of orthogonal polynomials of degree $n$.*

**Proof.** We go back to the generating function of $h$-harmonics in Definition 2.1. The explicit formula of $R_\alpha$ shows that $R_\alpha(x)$ is even in each of its variables only if each $\alpha_i$ is even for $i = 1, \ldots, d+1$. Let $\varepsilon \in \{-1, 1\}^{d+1}$. Then $R_\alpha(x\varepsilon) = R_\alpha(\varepsilon_1 x_1, \ldots, \varepsilon_{d+1} x_{d+1}) = \varepsilon^\alpha R_\alpha(x)$. It follows that

$$\sum_{\beta \in \mathbb{N}_0^{d+1}} b^{2\beta} R_{2\beta}(x) = \frac{1}{2^{d+1}} \sum_{\alpha \in \mathbb{N}_0^{d+1}} b^\alpha \sum_{\varepsilon \in \{-1,1\}^{d+1}} R_\alpha(x\varepsilon).$$

On the other hand, using the explicit formula of $V_\kappa$, the generating function gives

$$\frac{1}{2^{d+1}} \sum_{\varepsilon \in \{-1,1\}^{d+1}} \sum_{\alpha \in \mathbb{N}_0^{d+1}} b^\alpha R_\alpha(x\varepsilon)$$
$$= c_\kappa \int_{[-1,1]^{d+1}} \sum_{\varepsilon \in \{-1,1\}^{d+1}}$$
$$\times \frac{\prod_{i=1}^{d+1}(1 + t_i)(1 - t_i^2)^{\kappa_i - 1}}{(1 - 2(b_1 x_1 t_1 \varepsilon_1 + \cdots + b_{d+1} x_{d+1} t_{d+1} \varepsilon_{d+1}) + \|b\|^2)^\rho} \, dt$$

for $\|x\| = 1$. Changing variables $t_i \mapsto t_i \varepsilon_i$, the fact that $\sum_\varepsilon \prod_{i=1}^{d+1}(1 + \varepsilon_i t_i) = 2^{d+1}$ shows that the generating function of $R_{2\beta}(x)$ agrees with the generating function of $R_\beta^T(x_1^2, \ldots, x_{d+1}^2)$ in Definition 2.9. Consequently, the formulae of $R_\alpha^T$ follow from the corresponding ones for $R_{2\alpha}$. The polynomial $R_\alpha^T$ is homogeneous in $X$. Using the correspondence (2.9) between orthogonal polynomials on $S^d$ and on $T^d$, we see that $R_\alpha^T$ are orthogonal with respect to $W_\kappa^T$. If $\alpha_{d+1} = 0$, then $R_\alpha^T(x) = x^\alpha - Q_\alpha$, which proves the last statement of the proposition. □

In the case of $\alpha_{d+1} = 0$, the explicit formula of $R^T_\alpha$ shows that $R^T_{(\alpha,0)}(x) = x^\alpha - Q_\alpha(x)$; setting $b_{d+1} = 0$ in Definition 2.9 gives the generating function of $R^T_{(\alpha,0)}$. The explicit formula of $R^T_{(\alpha,0)}$ can be found in [7], which appeared earlier in the literature in some special cases. The generating function of $R^T_\alpha$ appears to be new in all cases. We note that if all $\kappa_i = 0$, then the integrals in the Definition 2.9 disappear, so that the generating function in the case of the Chebyshev weight function $W^T(x) = (x_1 \dots x_d(1 - |x|))^{-1/2}$ is simply $(1 - 2\langle b, x\rangle + \|b\|^2)^{-1}$.

## 3. Symmetric monomial orthogonal polynomials

Let $\mathcal{S}_{d+1}$ denote the symmetric group of $d+1$ objects. For a permutation $w \in \mathcal{S}_{d+1}$ we write $xw = (x_{w(1)}, \dots, x_{w(d+1)})$ and define $T(w)f(x) = f(xw)$. If $T(w)f = f$ for all $w \in \mathcal{S}_{d+1}$, we say that $f$ is invariant under $\mathcal{S}_{d+1}$.

For $\alpha \in \mathbb{N}_0^d$ and $w \in \mathcal{S}_{d+1}$, we define the action of $w$ on $\alpha$ by $(\alpha w)_i = \alpha_{w^{-1}(i)}$. Using this definition we have $(xw)^\alpha = x^{\alpha w}$.

### 3.1. *Symmetric monomial h-harmonics*

In this section, we assume that $\kappa_1 = \cdots = \kappa_{d+1} = \kappa$. Then the weight function $h_\kappa$ in (1.1) is invariant under $\mathcal{S}_{d+1}$. Let $\mathcal{H}_n^{d+1}(h_\kappa^2, \mathcal{S})$ denote the subspace of $h$-harmonics in $\mathcal{H}_n^{d+1}(h_\kappa^2)$ invariant under the group $\mathcal{S}_{d+1}$. Our goal is to give an explicit basis for $\mathcal{H}_n^{d+1}(h_\kappa^2, \mathcal{S})$.

A partition $\lambda$ of $d+1$ parts is an element in $\mathbb{N}_0^{d+1}$ such that $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_{d+1}$. Let $\Omega^{d+1}$ denote the set of partitions of $d+1$ parts. Let

$$\Omega_n^{d+1} = \{\lambda \in \mathbb{N}_0^{d+1} : \lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_{d+1}, |\lambda| = n\},$$

the set of $d+1$ parts partitions of size $n$, and let $\Lambda_n^{d+1} = \{\lambda \in \Omega_n : \lambda_1 = \lambda_2\}$. For a partition $\lambda$ the monomial symmetric polynomial $m_\lambda$ is defined by [14]

$$m_\lambda(x) = \sum \left\{x^\alpha : \alpha \text{ being distinct permutations of } \lambda\right\}.$$

Let $\mathcal{B}_{d+1}$ denote the hyperoctadedral group, which is a semi-direct product of $\mathbb{Z}_2^{d+1}$ and $\mathcal{S}_{d+1}$. A function $f$ is invariant under $\mathcal{B}_{d+1}$ if $f(x) = g(x_1^2, \dots, x_{d+1}^2)$ and $g$ is invariant under $\mathcal{S}_{d+1}$. Since the weight function $h_\kappa(x) = \prod_{i=1}^{d+1} |x_i|^\kappa$ is invariant under $\mathcal{B}_{d+1}$, the monomials $x^\alpha$ and $x^\beta$ are automatically orthogonal whenever $\alpha$ and $\beta$ are of different parity. Hence, closely related to $\mathcal{H}_n^{d+1}(h_\kappa^2, \mathcal{S})$ is the space $\mathcal{H}_n^{d+1}(h_\kappa^2, \mathcal{B})$, the subspace of $h$-harmonics in $\mathcal{H}_n^{d+1}(h_\kappa^2)$ invariant under $\mathcal{B}_{d+1}$. Recently, Dunkl [6] gave an explicit basis for $\mathcal{H}_n^{d+1}(h_\kappa^2, \mathcal{B})$ in the form of

$$\begin{aligned} p_\lambda(x) = m_\lambda(x^2) + \sum \{c_\mu m_\mu(x^2) : \mu \in \Omega_n^{d+1}, \\ \mu_i \leqslant \lambda_i, 2 \leqslant i \leqslant d+1, \mu \neq \lambda\}, \end{aligned} \tag{3.1}$$

where $x^2 = (x_1^2, \dots, x_{d+1}^2)$ and the coefficients $c_\mu$ were determined explicitly, and proved that the set $\{p_\lambda : \lambda \in \Lambda_n^{d+1}\}$ is a basis of $\mathcal{H}_n^{d+1}(h_\kappa^2, \mathcal{B})$.

Using the explicit formula of $R_\alpha$ we give a basis for $\mathcal{H}_n^{d+1}(h_\kappa^2, \mathcal{S})$ in this section. Let $\mathcal{S}_{d+1}(\lambda)$ denote the stabilizer of $\lambda$, $\mathcal{S}_{d+1}(\lambda) = \{w \in \mathcal{S}_{d+1} : \lambda w = \lambda\}$. Then we can write $m_\lambda = \sum x^{\lambda w}$ with the summation over all coset representatives of the subgroup $\mathcal{S}_{d+1}(\lambda)$ of $\mathcal{S}_{d+1}$, which we denote by $\mathcal{S}_{d+1}/\mathcal{S}_{d+1}(\lambda)$, it contains all $w$ such that $\lambda_i = \lambda_j$ and $i < j$ implies $w(i) < w(j)$.

**Definition 3.1.** Let $\lambda \in \Omega^{d+1}$. Define

$$S_\lambda(x) = \sum_{w \in \mathcal{S}_{d+1}/\mathcal{S}_{d+1}(\lambda)} R_{\lambda w}(x).$$

**Proposition 3.2.** *For $\lambda \in \Omega_n^{d+1}$, the polynomial $S_\lambda = \mathrm{proj}_n\, m_\lambda$ is an element of $\mathcal{H}_n^{d+1}(h_\kappa^2; \mathcal{S})$. Moreover, the set $\{S_\lambda : \lambda \in \Lambda_n^{d+1}\}$ is a basis of $\mathcal{H}_n^{d+1}(h_\kappa^2, \mathcal{S})$.*

**Proof.** The definition of $S_\lambda$ and the fact that $R_\alpha(x) = x^\alpha + \|x\|^2 Q_\alpha(x)$ shows

$$S_\lambda(x) = \sum_{w \in \mathcal{S}_{d+1}/\mathcal{S}_{d+1}(\lambda)} (x^{\lambda w} + \|x\|^2 Q_{\lambda w}(x)) = m_\lambda(x) + \|x\|^2 Q(x),$$

where $Q \in \Pi_{n-1}^{d+1}$. Also $S_\lambda \in \mathcal{H}_n^{d+1}(h_\kappa^2)$ since each $R_\alpha$ does. Hence, $S_\lambda = \mathrm{proj}_n\, m_\lambda$. It follows from (2) of Proposition 2.5 that,

$$S_\lambda(x) = \frac{(-1)^n}{2^n (\lambda)_n} \|x\|^{2\rho+2n} m_\lambda(\mathcal{D})(\|x\|^{-2\rho}),$$

which shows that $S_\lambda$ is symmetric. Since $\dim \mathcal{H}_n^{d+1}(h_\kappa^2, \mathcal{S}) = \#\Lambda_n^{d+1}$, we see that $\{S_\lambda : \lambda \in \Lambda_n^{d+1}\}$ is a basis of $\mathcal{H}_n^{d+1}(h_\kappa^2; \mathcal{S})$. $\square$

The fact that $S_\lambda$ is a symmetric polynomial also follows from a general statement about the best approximation by polynomials, proved in [1] for $L^p(S^d)$ and the proof carries over to the case $L^p(S^d; h_\kappa^2)$. Since the proof is short, we repeat it here. Let

$$\|f\|_p = \left(c_h' \int_{S^d} |f(y)|^p h_\kappa^2(y)\, d\omega(y)\right)^{1/p}$$

for $1 \leqslant p < \infty$ and let $\|f\|_\infty$ be the uniform norm on $S^d$.

**Proposition 3.3.** *If $f$ is invariant under $\mathcal{S}_{d+1}$ then the best approximation of $f$ in the space $L^p(S^d, h_\kappa^2)$ by polynomials of degree less than $n$ is attained by symmetric polynomials.*

**Proof.** Let $P \in \Pi_{n-1}^{d+1}$. Since $\kappa_1 = \cdots = \kappa_{d+1}$, $h_\kappa$ is invariant under the symmetric group, and so is the norms of the space $L^p(S^d; h_\kappa^2)$. Hence, the triangle inequality and the fact that $f$ is symmetric gives

$$\|f - P\|_p = \frac{1}{(d+1)!} \sum_{w \in \mathcal{S}_{d+1}} \|f(xw) - P(xw)\|_p$$

$$\geqslant \frac{1}{(d+1)!}\left\| \sum_{w\in\mathcal{S}_{d+1}} f(xw) - \sum_{w\in\mathcal{S}_{d+1}} P(xw)\right\|_p = \|f-P^*\|_p,$$

where $P^*$ is the symmetrization of $P$. Since $P^* \in \Pi_n^{d+1}$, this shows that the best approximation of $f$ can be attained by symmetric polynomials of the same degree. □

The best approximation in $L^2(S^d; h_\kappa^2)$ by polynomials is unique, so that a best approximation polynomial to a symmetric function must be a symmetric polynomial. Thus, the above proposition applies to $S_\lambda$, as $S_\lambda - m_\lambda$ is the best approximation to $m_\lambda$ in $L^2(S^d; h_\kappa^2)$ by polynomials of lower degrees.

From the definition of $S_\lambda$, it is not immediately clear that $S_\lambda$ is symmetric. Next, we give an explicit formula of $S_\lambda$ in terms of monomial symmetric functions and powers of $\|x\|$.

We start with the following simple observation:

**Lemma 3.4.** *Let $w \in \mathcal{S}_{d+1}$. Then $R_\alpha(xw) = R_{\alpha w}(x)$.*

**Proof.** This follows from the generating function of $R_\alpha(x)$. Indeed, let $\Phi_\kappa(t) = \prod_{i=1}^{d+1} c_k(1+t_i)(1-t_i^2)^{k-1}$; then $\Phi_\kappa(t)$ is invariant under $\mathcal{S}_{d+1}$. Hence, using the explicit formula of $V_\kappa$ in (2.2), it follows from the Definition 2.1 that

$$\begin{aligned}\sum b^\alpha \widetilde{R}_\alpha(xw) &= \int_{[-1,1]^{d+1}} \frac{1}{\left(1-2\sum b_i(xw)_i t_i + \|b\|^2\|x\|^2\right)^\rho}\,\Phi_\kappa(t)\,dt \\ &= \int_{[-1,1]^{d+1}} \frac{1}{\left(1-2\sum (bw^{-1})_i x_i t_i + \|b\|^2\|x\|^2\right)^\rho}\,\Phi_\kappa(t)\,dt \\ &= \sum (bw^{-1})^\alpha \widetilde{R}_\alpha(x) = \sum b^{\alpha w^{-1}} \widetilde{R}_\alpha(x) = \sum b^\alpha \widetilde{R}_{\alpha w}(x),\end{aligned}$$

since the sum is over all $\alpha \in \mathbb{N}_0^{d+1}$. □

We need one more definition. For any $\alpha \in \mathbb{N}_0^{d+1}$, let $\alpha^+$ be the unique partition such that $\alpha^+ = \alpha w$ for some $w \in \mathcal{S}_{d+1}$.

**Proposition 3.5.** *Let $\lambda \in \Omega_n^{d+1}$ and let $\rho = (d+1)\kappa + (d-1)/2$. Then*

$$S_\lambda(x) = m_\lambda(\mathbf{1}) \sum_{2\gamma\leqslant\lambda} a_{\lambda,\gamma}\|x\|^{2|\gamma|}\frac{m_{(\lambda-2\gamma)^+}(x)}{m_{(\lambda-2\gamma)^+}(\mathbf{1})}, \quad x \in \mathbb{R}^{d+1},$$

*where*

$$a_{\lambda,\gamma} = \frac{(-\lambda+[(\lambda+\mathbf{1})/2])_\gamma(-[(\lambda+\mathbf{1})/2]-\kappa+\mathbf{1}/2)_\gamma}{(-|\lambda|-\rho+1)_{|\gamma|}\gamma!}.$$

**Proof.** Let $d_\lambda = |\mathcal{S}_{d+1}(\lambda)|$. We can write $m_\lambda(x) = d_\lambda^{-1}\sum_{w\in\mathcal{S}_{d+1}} x^{\lambda w}$ and, using Lemma 3.4,

$$S_\lambda(x) = d_\lambda^{-1}\sum_{w\in\mathcal{S}_{d+1}} R_{\lambda w}(x) = d_\lambda^{-1}\sum_{w\in\mathcal{S}_{d+1}} R_\lambda(xw).$$

The coefficients $a_{\lambda,\gamma}$ appear in the explicit formula of $R_\lambda$. Indeed, from the formula in Proposition 2.3, $R_\lambda(x) = \sum a_{\lambda,\gamma} \|x\|^{2|\gamma|} x^{\lambda-2\gamma}$. For $w \in \mathcal{S}_{d+1}$ and $\lambda, \gamma \in \mathbb{N}^{d+1}$, we have $(\lambda w)_\gamma = (\lambda)_{\gamma w^{-1}}$. Therefore, as $|\alpha w| = |\alpha|$ for $\alpha \in \mathbb{N}_0^{d+1}$, it follows from the formula of $a_{\lambda,\gamma}$ that $a_{\lambda w,\gamma} = a_{\lambda,\gamma w^{-1}}$. Consequently,

$$\begin{aligned}\sum_{w \in S_{d+1}} R_\lambda(xw) &= \sum_{w \in S_{d+1}} \sum_\gamma a_{\lambda w,\gamma} \|x\|^{2|\gamma|} x^{\lambda w - 2\gamma} \\ &= \sum_{w \in S_{d+1}} \sum_\gamma a_{\lambda,\gamma w^{-1}} \|x\|^{2|\gamma|} x^{\lambda w - 2\gamma w^{-1} w} \\ &= \sum_\gamma a_{\lambda,\gamma} \|x\|^{2|\gamma|} \sum_{w \in S_{d+1}} x^{(\lambda-2\gamma)w},\end{aligned}$$

since the summation is over all $\gamma \in \mathbb{N}_0^d$. Note that the coefficients $a_{\lambda,\gamma} = 0$ if $\gamma_i > \lambda_i - [(\lambda_i+1)/2]$, so that $\lambda_i - 2\gamma_i \geqslant 0$. Therefore, we can write

$$\sum_{w \in S_{d+1}} x^{(\lambda-2\gamma)w} = \sum_{w \in S_{d+1}} x^{(\lambda-2\gamma)^+ w} = d_{(\lambda-2\gamma)^+} m_{(\lambda-2\gamma)^+}(x).$$

Put these formulae together, we get

$$S_\lambda(x) = d_\lambda^{-1} \sum_\gamma a_{\lambda,\gamma} \|x\|^{2|\gamma|} d_{(\lambda-2\gamma)^+} m_{(\lambda-2\gamma)^+}(x),$$

which gives the stated formula upon using the fact that $m_\lambda(\mathbf{1}) = (d+1)!/d_\lambda$. □

In the simplest case of $\lambda = (n, 0, \ldots, 0) = ne_1$, we conclude that

$$\begin{aligned}S_{ne_1}(x) &= \sum_j \frac{(-n + [\frac{n+1}{2}])_j (-[\frac{n+1}{2}] - \kappa + \frac{1}{2})_j}{(-n-\rho+1)_j\, j!} \|x\|^{2j} m_{(n-2j)e_1}(x) \\ &= \|x\|^n \sum_{i=1}^{d+1} \left[k_n^{(\rho-\kappa,\kappa)}\right]^{-1} C_n^{(\rho-\kappa,\kappa)}(x_i/\|x\|),\end{aligned} \tag{3.2}$$

where $\rho = (d+1)\kappa + (d-1)/2$, and the second equality follows from the definition of $C_n^{(\lambda,\mu)}$ or from Corollary 2.6.

Since the sum in the formula of $S_\lambda$ is over all $\gamma \in \mathbb{N}_0^{d+1}$, some $m_\mu$ may appear several times in the sum. With a little more effort one may write $S_\lambda$ in a more compact form. Evidently, this depends on how many parts of $\lambda$ are repeated. We shall consider only a simple case of $\lambda = (q, \ldots, q)$, in which all parts are equal.

**Corollary 3.6.** *For* $\lambda = (q, \ldots, q)$, $q \in \mathbb{N}_0$,

$$S_\lambda(x) = \sum_{\mu \in \Omega^{d+1}} a_{\lambda,\mu} \|x\|^{2|\mu|} m_{\lambda-2\mu}(x).$$

**Proof.** In this case, it is easy to see that $a_{\lambda,\mu w} = a_{\lambda,\mu}$ for each $w \in \mathcal{S}_{d+1}$. Moreover, $d_\lambda = (d+1)!$. Consequently, using the fact that $\sum_\gamma c_\gamma = \sum_{\mu \in \Omega^{d+1}} d_\mu^{-1} \sum_{w \in \mathcal{S}_{d+1}} c_{\mu w}$, it follows that

$$
\begin{aligned}
S_\lambda(x) &= \frac{1}{(d+1)!} \sum_\gamma a_{\lambda,\gamma} \|x\|^{2|\gamma|} d_{(\lambda-2\gamma)^+} m_{(\lambda-2\gamma)^+}(x) \\
&= \frac{1}{(d+1)!} \sum_{\mu\in\Omega^{d+1}} d_\mu^{-1} \sum_{w\in\mathcal{S}_{d+1}} a_{\lambda,\mu w} \|x\|^{2|\mu|} d_{(\lambda-2\mu w)^+} m_{(\lambda-2\mu w)^+}(x) \\
&= \sum_{\mu\in\Omega^{d+1}} a_{\lambda,\mu} \|x\|^{2|\mu|} \frac{1}{(d+1)!} \sum_{w\in\mathcal{S}_{d+1}} m_{(\lambda-2\mu w)^+}(x),
\end{aligned}
$$

since $\lambda = (q, \ldots, q)$ implies that $d_{(\lambda-2\mu w)^+} = d_{\mu w} = d_\mu$. Also, the special form of $\lambda$ implies $m_{(\lambda-2\mu w)^+} = m_{\lambda-2(\mu w)^+} = m_{\lambda-2\mu}$, which completes the proof. □

Since $m_{2\lambda}(x) = m_\lambda(x_1^2, \ldots, x_{d+1}^2)$, the theorem shows that the set $\{S_{2\lambda} : \lambda \in \Lambda_n^{d+1}\}$ is a basis for the space $\mathcal{H}_n^{d+1}(h_\kappa^2; \mathcal{B})$. These results are interesting even in the case of the ordinary harmonics ($\kappa = 0$). The only other symmetric orthogonal basis known is given by Dunkl [6] recently for $\mathcal{H}_n^{d+1}(h_\kappa^2; \mathcal{B})$. It should be pointed out, however, that $S_\lambda$ are not mutually orthogonal for $\lambda \in \Omega_n^{d+1}$. We do not know how to construct an orthonormal basis for $\mathcal{H}_n^{d+1}(h_\kappa^2; \mathcal{S})$ or if there is a compact formula for the $L^2$ norm of $S_\lambda$.

Since $\|x\|^2$ is symmetric, one can write $\|x\|^2 m_\mu$ in terms of symmetric monomial polynomials $m_\sigma$ so that $S_{2\lambda}$ can be written in terms of $m_\mu(x^2)$ as in Dunkl's basis (3.1). It turns out, however, that the two bases $\{S_{2\lambda} : \lambda \in \Lambda_n^{d+1}\}$ and $\{p_\lambda : \lambda \in \Lambda_n^{d+1}\}$ are quite different and they are in fact biorthogonal [6].

### 3.2. *Symmetric monomial orthogonal polynomials on the unit ball*

On the unit ball $B^d$ we consider the weight function $W_\kappa^B(x)$ with $\kappa_1 = \cdots = \kappa_d = 0$. Writing $\kappa_{d+1} = \mu$, we write $W_{\kappa,\mu}^B$ instead of $W_\kappa^B$. That is,

$$
W_{\kappa,\mu}^B(x) = \prod_{i=1}^d |x_i|^{2\kappa} (1 - \|x\|^2)^{\mu-1/2}, \quad x \in B^d.
$$

This weight function is evidently invariant under the symmetric group $\mathcal{S}_d$. Let $\mathcal{V}_n^d(W_{\kappa,\mu}^B; \mathcal{S})$ denote the space of symmetric orthogonal polynomials of degree $n$ with respect to $W_{\kappa,\mu}^B$. The dimension of this space is dim $\mathcal{V}_n^d(W_{\kappa,\mu}^B; \mathcal{S}) = \#\Omega_n^d$, the cardinality of $d$-parts partitions of size $n$, since a basis can be obtained by applying Gram–Schmidt process on a basis of symmetric polynomials of degree at most $n$ in $d$ variables.

For symmetric orthogonal polynomials we cannot use the correspondence (2.8) between $h$-harmonics and orthogonal polynomials on the unit ball, since $R_\alpha^B(x) = R_{(\alpha,0)}(x, x_{d+1})$. On the other hand, the polynomial $R_\alpha^B$ in Definition 2.7 is similar to $R_\alpha^B$ specified in Definition 2.1. The similarity allows us to carry out the study in the previous subsection with little additional effort. We define $S_\lambda^B$ as in Definition 3.1:

**Definition 3.7.** Let $\lambda \in \Omega^d$. Define

$$S_\lambda^B(x) = \sum_{w \in \mathcal{S}_{d+1}/\mathcal{S}_d(\lambda)} R_{\lambda w}^B(x).$$

**Proposition 3.8.** *For $\lambda \in \Omega_n^d$, the polynomial $S_\lambda$ is the orthogonal projection of the symmetric monomial polynomial $m_\lambda$ onto $\mathcal{V}_n^d(W_{\kappa,\mu}^B; \mathcal{S})$. Moreover, the set $\{S_\lambda : \lambda \in \Omega_n^d\}$ is a basis of $\mathcal{V}_n^d(W_{\kappa,\mu}^B; \mathcal{S})$.*

We again have $R_{\alpha w}^B(x) = R_\alpha^B(xw)$ for any $w \in \mathcal{S}_d$ and we can derive an explicit formula of $S_\lambda^B$ as in Proposition 3.5:

**Proposition 3.9.** *Let $\lambda \in \Omega_n^d$ and let $\rho = d\kappa + \mu + (d-1)/2$. Then*

$$S_\lambda^B(x) = m_\lambda(\mathbf{1}) \sum_{2\gamma \leqslant \lambda} a_{\lambda,\gamma} \frac{m_{(\lambda-2\gamma)^+}(x)}{m_{(\lambda-2\gamma)^+}(\mathbf{1})}, \quad x \in \mathbb{R}^d,$$

*where $\mathbf{1} = (1, \ldots, 1) \in \mathbb{N}_0^d$ and*

$$a_{\lambda,\gamma} = \frac{(-\lambda + [(\lambda+\mathbf{1})/2])_\gamma(-[(\lambda+\mathbf{1})/2] - \kappa + \mathbf{1}/2)_\gamma}{(-|\lambda| - \rho + 1)_{|\gamma|}\gamma!}.$$

In the simplest case of $\lambda = (n, 0, \ldots, 0) = ne_1 \in \mathbb{R}^d$, we conclude that

$$\begin{aligned} S_{ne_1}^B(x) &= \sum_j \frac{(-n + [\frac{n+1}{2}])_j(-[\frac{n+1}{2}] - \kappa_1 + \frac{1}{2})_j}{(-n-\rho+1)_j j!} m_{(n-2j)e_1}(x) \\ &= \sum_{i=1}^d \left[k_n^{(\rho-\kappa_1,\kappa_1)}\right]^{-1} C_n^{(\rho-\kappa_1,\kappa_1)}(x_1), \end{aligned}$$

where $\rho = d\kappa + \mu + (d-1)/2$, since $R_{ne_1}^B(x) = R_{ne_1'}(x, x_{d+1})$ for $(x, x_{d+1}) \in S^d$, where $e_1' = (1, 0, \ldots, 0) = (e_1, 0) \in \mathbb{R}^{d+1}$, and Corollary 2.6 shows that $R_{ne_i}^B(x) = \left[k_n^{(\rho-\kappa_i,\kappa_i)}\right]^{-1} C_n^{(\rho-\kappa_i,\kappa_i)}(x_i)$.

### 3.3. *Symmetric monomial orthogonal polynomials on the simplex*

We can also give explicit formulae for the symmetric monomial orthogonal polynomials with respect to $W_\kappa^T$ on the simplex.

On the simplex $T^d$, it is natural to consider the symmetric group $\mathcal{S}_{d+1}$ of the vertices of $T^d$. A function $f(x)$ on $T^d$ is symmetric if in the homogeneous coordinates $X = (x, x_{d+1})$, $x_{d+1} = 1 - |x|$, $f(x) = g(X)$ is invariant under $\mathcal{S}_{d+1}$; that is, if $g(Xw) = g(X)$ for every $w \in \mathcal{S}_{d+1}$. Let $\mathcal{V}_n^{d+1}(W_\kappa^T; S)$ denote the space of orthogonal polynomials of degree $n$ that are symmetric.

**Proposition 3.10.** *For each $\lambda \in \Omega_n^{d+1}$, the polynomial*

$$S_\lambda^T(x) = \sum_{w \in \mathcal{S}_{d+1}/\mathcal{S}_{d+1}(\lambda)} R_{\lambda w}^T(x)$$

*is a symmetric orthogonal polynomial and $S_\lambda^T(x) = m_\lambda(X) + Q(x)$, $Q \in \Pi_{n-1}^{d+1}$. Moreover,*

$$S_\lambda^T(x) = m_\lambda(\mathbf{1}) \sum_\gamma \frac{(-\lambda)_\gamma(-\lambda-\kappa+\mathbf{1}/2)_\gamma}{(-2|\lambda|-\rho+1)_{|\gamma|}\gamma!} \cdot \frac{m_{(\lambda-\gamma)^+}(X)}{m_{(\lambda-\gamma)^+}(\mathbf{1})},$$

*where $\mathbf{1} = (1, \ldots, 1) \in \mathbb{N}_0^{d+1}$. Furthermore, the set $\{S_\lambda^T : \lambda \in \Lambda_n^{d+1}\}$ is a basis of $\mathcal{V}_n^{d+1}(W_\kappa^T; S)$.*

This follows from the correspondence (2.9) and the properties of $S_{2\lambda}$. Notice that $m_{2\lambda}(x) = m_\lambda(x_1^2, \ldots, x_{d+1}^2)$.

## 4. Norm of the monomial polynomials

### 4.1. *Norm of monomial h-harmonics*

Since $R_\alpha$ is orthogonal to polynomials in $\Pi_{n-1}^{d+1}$ with respect to $h_\kappa^2\, d\omega$ on $S^d$ and $R_\alpha(x) - x^\alpha$ is a polynomial of lower degree when restricted to $S^d$, the standard Hilbert space theory shows that the polynomial $R_\alpha$ is the best approximation of $x^\alpha$ in the $L^2$ norm defined by

$$\|f\|_2 = \left(c_h' \int_{S^d} |f(y)|^2 h_\kappa^2(y)\, d\omega(y)\right)^{1/2},$$

where $c_h'$ is the normalization constant of $h_\kappa^2$. In other words, the polynomial $x^\alpha - R_\alpha$ has the smallest $L^2$ norm among all polynomials of the form $x^\alpha - P(x)$, $P \in \Pi_{n-1}^{d+1}$ on $S^d$. That is,

$$\|R_\alpha\|_2 = \min_{P \in \Pi_{n-1}^{d+1}} \|x^\alpha - P\|_2, \quad |\alpha| = n.$$

In the following we compute the $L^2$ norm of $R_\alpha$.

**Theorem 4.1.** *Let $\rho = |\kappa| + \frac{d-1}{2}$. Let $\alpha \in \mathbb{N}_0^{d+1}$ and denote $\beta = \alpha - [(\alpha + \mathbf{1})/2]$. Then*

$$c_h' \int_{S^d} |R_\alpha(x)|^2 h_\kappa^2(x)\, d\omega = \frac{\rho\,(\kappa + \mathbf{1}/2)_\alpha}{(\rho)_{|\alpha|}} \sum_\gamma \frac{(-\beta)_\gamma \left(-\alpha + \beta - \kappa + \mathbf{1}/2\right)_\gamma}{(-\alpha - \kappa + \mathbf{1}/2)_\gamma\, \gamma!(|\alpha| - |\gamma| + \rho)}$$

$$= 2\rho \frac{\beta!\,(\kappa + \mathbf{1}/2)_{\alpha-\beta}}{(\rho)_{|\alpha|}} \int_0^1 \prod_{i=1}^{d+1} C_{\alpha_i}^{(\frac{1}{2},\kappa_i)}(t)\, t^{|\alpha|+2\rho-1}\, dt.$$

**Proof.** Using the explicit formula of $R_\alpha(x)$ and the Beta-type integral,

$$c_h' \int_{S^d} x^{2\sigma} h_\kappa^2(x)\, d\omega = \frac{\Gamma(|\kappa| + (d+\mathbf{1})/2)}{\Gamma(|\sigma| + |\kappa| + (d+1)/2)} \prod_{i=1}^{d+1} \frac{\Gamma(\sigma_i + \kappa_i + 1/2)}{\Gamma(\kappa_i + 1/2)} = \frac{(\kappa + \mathbf{1}/2)_\sigma}{(\rho + \mathbf{1})_{|\sigma|}},$$

it follows from the explicit formula of $R_\alpha$ in Proposition 2.3 that

$$c_h' \int_{S^d} |R_\alpha(x)|^2 h_\kappa^2(x)\, d\omega = c_h' \int_{S^d} R_\alpha(x) x^\alpha h_\kappa^2(x)\, d\omega = \sum_\gamma \frac{(-\beta)_\gamma (-\alpha + \beta - \kappa + \mathbf{1}/2)_\gamma (\kappa + \mathbf{1}/2)_{\alpha-\gamma}}{(-|\alpha| - \rho + 1)_{|\gamma|} \gamma! (\rho + \mathbf{1})_{|\alpha|-|\gamma|}}.$$

Rewriting the sum using $(a)_{n-m} = (-1)^m (a)_n/(1-n-a)_m$ and $(-a)_n/(-a+1)_n = a/(a-n)$ gives the first stated equation. To derive the second equation, we show that the sum in the first equation can be written as an integral. We define a function

$$F(r) = \sum_{\gamma \leqslant \beta} \frac{(-\beta)_\gamma (-\alpha + \beta - \kappa + \mathbf{1}/2)_\gamma}{(-\alpha - \kappa + \mathbf{1}/2)_\gamma \gamma! (|\alpha| - |\gamma| + \rho)} r^{|\alpha| - |\gamma| + \rho}.$$

Evidently, $F(1)$ is the sum in the first equation. Moreover, the sum is a finite sum over $\gamma \leqslant \beta$ as $(-\beta)_\gamma = 0$ for $\gamma > \beta$, it follows that $F(0) = 0$. Hence, the sum $F(1)$ is given by $F(1) = \int_0^1 F'(r)\, dr$. The derivative of $F$ can be written as

$$\begin{aligned} F'(r) &= \sum_\gamma \frac{(-\beta)_\gamma (-\alpha + \beta - \kappa + \mathbf{1}/2)_\gamma}{(-\alpha - \kappa + \mathbf{1}/2)_\gamma \gamma!} r^{|\alpha| - |\gamma| + \rho - 1} \\ &= r^{|\alpha| + \rho - 1} \prod_{i=1}^{d+1} \sum_{\gamma_i} \frac{(-\beta_i)_{\gamma_i} (-\alpha_i + \beta_i - \kappa_i + 1/2)_{\gamma_i}}{(-\alpha_i - \kappa_i + 1/2)_{\gamma_i} \gamma_i!} r^{-\gamma_i} \\ &= r^{|\alpha| + \rho - 1} \prod_{i=1}^{d+1} {}_2F_1 \left( \begin{matrix} -\beta_i, -\alpha_i + \beta_i - \kappa_i + 1/2 \\ -\alpha_i - \kappa_i + 1/2 \end{matrix}; \frac{1}{r} \right). \end{aligned}$$

The Jacobi polynomial $P_n^{(a,b)}$ can be written as ${}_2F_1$ in a different form [15, (4.22.1)],

$$P_n^{(a,b)}(t) = \binom{2n + a + b}{n} \left( \frac{t-1}{2} \right)^n {}_2F_1 \left( \begin{matrix} -n, -n-a \\ -2n - a - b \end{matrix}; \frac{2}{1-t} \right).$$

Use this formula with $n = \beta_i$, $a = \alpha_i - 2\beta_i + \kappa_i - \frac{1}{2}$, $b = 0$ and $r = (1-t)/2$, and then use $P_n^{(a,b)}(t) = (-1)^n P_n^{(b,a)}(-t)$, we conclude

$$F'(r) = \frac{(\kappa + \mathbf{1}/2)_{\alpha - \beta} \beta!}{(\kappa + \mathbf{1}/2)_\alpha} r^{|\alpha| - |\beta| + \rho - 1} \prod_{i=1}^{d+1} P_{\beta_i}^{(0, \alpha_i - 2\beta_i + \kappa_i - 1/2)}(2r - 1).$$

Consequently, it follows that

$$F(1) = \frac{(\kappa + \mathbf{1}/2)_{\alpha-\beta}\beta!}{(\kappa + \mathbf{1}/2)_{\alpha}} \int_0^1 \prod_{i=1}^{d+1} P_{\beta_i}^{(0,\alpha_i-2\beta_i+\kappa_i-1/2)}(2r-1) r^{|\alpha|-|\beta|+\rho-1}\, dr. \quad (4.1)$$

From the relation (2.5) it follows that $P_{\beta_i}^{(0,\alpha_i-2\beta_i+\kappa_1-1/2)}(2t^2-1) = C_{\alpha}^{(1/2,\kappa_i)}(t)$ if $\alpha_i$ is even, and $t P_{\beta_i}^{(0,\alpha_i-2\beta_i+\kappa_1-1/2)}(2t^2-1) = C_{\alpha}^{(1/2,\kappa_i)}(t)$ if $\alpha_i$ is odd. Hence, changing variables $r \to t^2$ in the above integral leads to the second stated equation. □

The constant in the second equal sign can be written in terms of $k_n^{(1/2,\kappa_i)}$, the leading coefficient of $C_n^{(1/2,\kappa_i)}$, by using (2.7) and considering $\alpha_i$ being even and odd separately. As an equivalent statement, the theorem gives

**Corollary 4.2.** *Let* $\alpha \in \mathbb{N}_0^d$ *and* $n = |\alpha|$. *Then*

$$\inf_{Q \in \Pi_{n-1}^d} \|x^\alpha - Q(x)\|_2^2 = \frac{2\rho\left(\kappa + \frac{1}{2}\right)_\alpha}{(\rho)_{|\alpha|}} \int_0^1 \prod_{i=1}^{d+1} \frac{C_{\alpha_i}^{(\frac{1}{2},\kappa_i)}(t)}{k_{\alpha_i}^{(\frac{1}{2},\kappa_i)}} t^{|\alpha|+2\rho-1}\, dt.$$

In the case of $d = 1$, the integral contains the product of two Jacobi polynomials. Moreover, the parameters satisfy a condition for which the integral can be written as a terminating ${}_3F_2$ and simplified by the known formula (see [9, vol. 2, p. 286])

$$\int_0^1 P_{\beta_1}^{(0,\sigma_1)}(2r-1) P_{\beta_2}^{(0,\sigma_2)}(2r-1) r^{|\beta|+|\sigma|}\, dr$$
$$= \frac{(|\beta|)!(|\sigma|+1)_{|\beta|}(\sigma_1+1)_{|\beta|}(\sigma_2+1)_{|\beta|}}{(|\sigma|+2)_{2|\beta|}(\sigma_1+1)_{\beta_1}(\sigma_2+1)_{\beta_2}(|\sigma|+2|\beta|+1)}.$$

Using this formula with an obvious choice of the parameters, the norm of $R_\alpha$ for $d = 1$ can be written in a compact form. Equivalently, this gives

**Corollary 4.3.** *Let* $\alpha = (\alpha_1, \alpha_2)$ *and write* $\sigma = [(\alpha + 1)/2]$. *Then*

$$\inf_{Q \in \Pi_{n-1}^2} \|x^\alpha - Q(x)\|_2^2 = \frac{(|\kappa|)_{|\sigma|}(|\alpha| - |\sigma|)!}{(|\kappa|+1)_{|\alpha|}(|\kappa|)_{|\alpha|}} \times \left(\kappa_1 + \frac{1}{2}\right)_{\sigma_1+\alpha_2-\sigma_2} \left(\kappa_2 + \frac{1}{2}\right)_{\sigma_2+\alpha_1-\sigma_1}.$$

For $d > 1$ and $\alpha = ne_i$, the sum in Theorem 4.1 is a balanced ${}_3F_2$, which can be summed using the Saalschütz summation formula. Alternatively, we can evaluate the norm of $R_{ne_i}$ by using the explicit formula of $R_{ne_i}$ in Corollary 2.6 and the formula

$$\int_{S^d} f(x)\, d\omega_d = \int_0^\pi \int_{S^{d-1}} f(\cos\theta, \sin\theta x')\, d\omega_{d-1}(x')(\sin\theta)^{d-1}\, d\theta. \quad (4.2)$$

This way, the norm of $R_{ne_i}$ can be derived from the leading coefficient $k_n^{(\lambda,\mu)}$, given in (2.7), of $C_n^{(\lambda,\mu)}$ and the norm of $C_n^{(\lambda,\mu)}(t)$. We denote by $h_n^{(\lambda,\mu)}$ the $L^2$ norm of $C_n^{(\lambda,\mu)}$ with respect to the normalized weight function $c_{\lambda,\mu} w_{\lambda,\mu}(t)$, where $w_{\lambda,\mu}(t) = |t|^{2\mu}(1-t^2)^{\lambda-1/2}$ and $c_{\lambda,\mu}^{-1} = \Gamma(\mu+1/2)\Gamma(\lambda+1/2)/\Gamma(\lambda+\mu+1)$. It is given by [7, p. 27]

$$h_{2m}^{(\lambda,\mu)} = \frac{(\lambda+\frac{1}{2})_m(\lambda+\mu)_m(\lambda+\mu)}{m!(\mu+\frac{1}{2})_m(\lambda+\mu+2m)},$$
$$h_{2m+1}^{(\lambda,\mu)} = \frac{(\lambda+\frac{1}{2})_m(\lambda+\mu)_{m+1}(\lambda+\mu)}{m!(\mu+\frac{1}{2})_{m+1}(\lambda+\mu+2m+1)}. \tag{4.3}$$

We will follow the second approach to evaluate the norm of $R_{ne_i}$ since an intermediate result will be used later in the section.

**Corollary 4.4.** *For $n \in \mathbb{N}_0^d$, let $m = [(n+1)/2]$. Then*

$$\inf_{Q \in \Pi_{n-1}^d} \|x_i^n - Q(x)\|_2^2 = \frac{(n-m)!\left(\kappa_i+\frac{1}{2}\right)_n \left(|\kappa|-\kappa_i+\frac{d}{2}\right)_{n-m}}{(|\kappa|+\frac{d+1}{2})_n \left(m+|\kappa|+\frac{d-1}{2}\right)_{n-m}\left(m+\kappa_i+\frac{1}{2}\right)_{n-m}}.$$

**Proof.** We only need to prove the case $i=1$. For $x \in S^d$, write $x = (\cos\theta x', \sin\theta)$, $x' \in S^{d-1}$. Let $\lambda_1 = \rho - \kappa_1 = |\kappa| - \kappa_1 + (d-1)/2$. Using the explicit formula of $R_{ne_i}$ in Corollary 2.6, Eq. (4.2) with a change of variable $t = \cos\theta$ shows that

$$c_h' \int_{S^d} |R_{ne_1}(t)|^2\, dt = c_h' \int_{-1}^1 \left|\frac{C_n^{(\lambda_1,\kappa_1)}(t)}{k_n^{(\lambda_1,\kappa_1)}}\right|^2 w_{\lambda_1,\kappa_1}(t)\, dt$$
$$\times \int_{S^{d-1}} \prod_{i=2}^{d+1} |x_i'|^{2\kappa_i}\, d\omega_{d-1}(x')$$
$$= h_n^{(\lambda_1,\kappa_1)} / \left[k_n^{(\lambda_1,\kappa_1)}\right]^2.$$

Hence, the stated formula follows from the explicit formulae of $k_n^{(\lambda,\kappa)}$ in (2.7) and $h_n^{(\lambda,\kappa)}$ in (4.3). □

We note that Corollary 4.2 and the above proof implies the formula

$$\frac{2\rho(\kappa_1+\frac{1}{2})_n}{(\rho)_n} \int_0^1 \frac{C_n^{(\frac{1}{2},\kappa_1)}(t)}{k_n^{(\frac{1}{2},\kappa_1)}} t^{|\alpha|+2\rho-1}\, dt = c_{\lambda_1,\mu_1} \int_{-1}^1 \left|\frac{C_n^{(\lambda_1,\kappa_1)}(t)}{k_n^{(\lambda_1,\kappa_1)}}\right|^2 w_{\lambda_1,\kappa_1}(t)\, dt,$$

which does not seem to follow from a simple transformation. This suggests the possibility that the norm of $R_\alpha$ may be expressed in some other, perhaps more illuminating, ways.

In general, however, the norm of $R_\alpha$ may not have a compact formula in the form of a ratio of products of Pochhammer symbols. For example, if $\alpha = (\alpha_1, \alpha_2, 0, \ldots, 0)$, then the integral in Theorem 4.1 becomes (see (4.1))

$$I(\sigma,\beta) := \int_0^1 P_{\beta_1}^{(0,\sigma_1)}(2r-1) P_{\beta_2}^{(0,\sigma_1)}(2r-1) r^{\sigma_1+\sigma_2+\beta_1+\beta_2+a}\, dr \tag{4.4}$$

with $\sigma_i = \alpha_i - 2\beta_i + \kappa_i - 1/2$, $\beta_i = \alpha_i - [(\alpha_i+1)/2]$ and $a = |\kappa| - \kappa_1 - \kappa_2 + (d-1)/2$. Using the ${}_2F_1$ formula of the Jacobi polynomials, this integral can be written as a single sum of a balanced ${}_4F_3$ series evaluated at 1,

$$I(\sigma,\beta) = \frac{(-1)^{\beta_1}(\sigma_1+1)_{\beta_1}(\sigma_1+a+1)_{|\beta|}}{\beta_1!(|\beta|+|\sigma|+a+2)(|\beta|+|\sigma|+a+2)_{\beta_2}(\sigma_1+a+1)_{\beta_1}} \times {}_4F_3\left(\begin{matrix} -\beta_1, \beta_1+\sigma_1+1, |\beta|+|\sigma|+a+1, |\beta|+\sigma_1+a+1 \\ |\sigma|+|\beta|+\beta_2+a+2, \sigma_1+1, \sigma_1+\beta_1+a+1 \end{matrix}; 1\right). \tag{4.5}$$

This ${}_4F_3$ is a finite sum, but it does not seem to have a compact form.

As a consequence of Theorem 4.1, the integral of the product generalized Gegenbauer polynomials in the theorem is positive, which does not seem to be obvious. It shows, in particular, that the expression $I(\sigma,\beta)$ is positive if $\sigma_i \geqslant 0$, $\alpha_i \geqslant 0$ and $a \geqslant 0$.

For the symmetric orthogonal polynomials, there is one simple case for which we can compute the norm explicitly, the norm of the symmetric polynomials $S_{ne_1}$ in (3.2). Recall that by Corollary 2.6 and (3.2), $S_{ne_1} = R_{ne_1} + \cdots + R_{ne_{d+1}}$ when $\kappa_i = \kappa$ for $1 \leqslant i \leqslant d+1$, and $R_{ne_i}$ is given in terms of $C_n^{(\rho-\kappa_i,\kappa_i)}(t)$. The key ingredient is the lemma below.

**Lemma 4.5.** *Let $\lambda_i = |\kappa| - \kappa_i + \frac{d-1}{2}$. Then for $n = 2m$,*

$$c_h' \int_{S^d} \frac{C_n^{(\lambda_1,\kappa_1)}(x_1) C_n^{(\lambda_2,\kappa_2)}(x_2)}{k_n^{(\lambda_1,\kappa_1)} k_n^{(\lambda_2,\kappa_2)}} h_\kappa^2(x)\, d\omega = (-1)^m \frac{(\kappa_2+\frac12)_m}{(\lambda_1+\frac12)_m} \frac{h_n^{(\lambda_1,\kappa_1)}}{\left[k_n^{(\lambda_1,\kappa_1)}\right]^2}.$$

**Proof.** For $x \in S^d$, write $x = (\cos\theta, \sin\theta x')$, $x' \in S^{d-1}$ and $0 \leqslant \theta \leqslant \pi$. Using the integration formula (4.2) and changing variable $t = \cos\theta$, we see that the left-hand side of the stated integral is equal to

$$c_h' \int_{-1}^1 \frac{C_n^{(\lambda_1,\kappa_1)}(t)}{k_n^{(\lambda_1,\kappa_1)}} \left[ \int_{S^{d-1}} \frac{C_n^{(\lambda_2,\kappa_2)}(\sqrt{1-t^2}x')}{k_n^{(\lambda_2,\kappa_2)}} \prod_{i=2}^{d+1} |x_i'|^{2\kappa_i}\, d\omega_{d-1}(x') \right] \times |t|^{2\kappa_1}(1-t^2)^{\lambda_1-\frac12}.$$

Since $n = 2m$, the integral inside the square bracket is a polynomial of degree $n$ in $t$ whose leading term is $(1-t^2)^m = (-1)^m t^{2m} + \cdots$. Consequently, by the orthogonality of $C_n^{(\lambda_1,\kappa_1)}(t)$, it follows that the above integral is equal to

$$\begin{aligned} &(-1)^m c_h' \int_{-1}^1 \frac{C_n^{(\lambda_1,\kappa_1)}(t)}{k_n^{(\lambda_1,\kappa_1)}} t^{2m} |t|^{2\kappa_1}(1-t^2)^{\lambda_1-\frac12}\, dt \\ &\quad \times \int_{S^{d-1}} {x_2'}^{2m} \prod_{i=2}^{d+1} |x_i'|^{2\kappa_i}\, d\omega_{d-1}(x') \\ &\quad = (-1)^m \frac{(\kappa_2+\frac12)_m}{(\lambda_1+\frac12)_m} \frac{1}{\left[k_n^{(\lambda_1,\kappa_1)}\right]^2} c_{\lambda_1,\kappa_1} \int_{-1}^1 |C_n^{(\lambda_1,\kappa_1)}(t)|^2 w_{\lambda_1,\kappa_1}(t)\, dt \end{aligned}$$

using (2.4), which gives the stated formula. □

Together with the proof of Corollary 4.4, this lemma allows us to compute the norm of any linear combination $b_1 R_{ne_1} + \cdots + b_{d+1} R_{ne_{d+1}}$, without the need of assuming $\kappa_i = \kappa$ for all $i$. We shall, however, use it only in the case of $\kappa_1 = \cdots = \kappa_{d+1}$ to compute the norm of $S_{ne_1}$. The proof shows clearly how the norm of the general case can be computed.

**Proposition 4.6.** *Let $\kappa_1 = \cdots = \kappa_{d+1} = \kappa$. Let $\lambda = d\kappa + \frac{d-1}{2}$. Then for $n = 2m$,*

$$\inf_{Q \in \Pi_{n-1}^{d+1}} \|x_1^n + \cdots + x_{d+1}^n - Q(x)\|_2^2 = (d+1)\left(1 + d(-1)^m \frac{(\kappa+\frac{1}{2})_m}{(\lambda+\frac{1}{2})_m}\right) \times \frac{(\lambda+\frac{1}{2})_m(\lambda+\kappa)_m(\kappa+\frac{1}{2})_m m!}{(\lambda+\kappa)_{2m}(\lambda+\kappa+1)_{2m}}$$

*and for $n = 2m+1$,*

$$\inf_{Q \in \Pi_{n-1}^{d+1}} \|x_1^n + \cdots + x_{d+1}^n - Q(x)\|_2^2 = (d+1)\frac{(\lambda+\frac{1}{2})_m(\lambda+\kappa)_{m+1}(\kappa+\frac{1}{2})_{m+1} m!}{(\lambda+\kappa)_{2m+1}(\lambda+\kappa+1)_{2m+1}}.$$

*In particular, the case $\kappa = 0$ and $\lambda = (d-1)/2 > 0$ gives the best approximation of $S_{ne_1}$ in the $L^2$ norm with respect to the surface measure $d\omega$.*

**Proof.** Since $|\kappa| = (d+1)\kappa$, $\lambda_i$ in the lemma becomes $\lambda = d\kappa + (d-1)/2$. By (3.2), for $\|x\| = 1$, we have

$$\begin{aligned}
&\int_{S^d} \left[S_{ne_1}(x)\right]^2 h_\kappa^2(x)\, d\omega \\
&= \int_{S^d} \left[\sum_{i=1}^{d+1} \left[k_n^{(\lambda,\kappa)}\right]^{-1} C_n^{(\lambda,\kappa)}(x_i)\right]^2 h_\kappa^2(x)\, d\omega \\
&= \sum_{i=1}^{d+1} \int_{S^d} \left[\frac{C_n^{(\lambda,\kappa)}(x_i)}{k_n^{(\lambda,\kappa)}}\right]^2 h_\kappa^2(x)\, d\omega \\
&\quad + \sum_{i \neq j} \int_{S^d} \frac{C_n^{(\lambda,\kappa)}(x_i) C_n^{(\lambda,\kappa)}(x_j)}{k_n^{(\lambda,\kappa)} k_n^{(\lambda,\kappa)}} h_\kappa^2(x)\, d\omega.
\end{aligned}$$

If $n = 2m+1$, then the integrals in the second sum is zero since $C_{2m+1}^{(\lambda,\mu)}(t)$ is an odd polynomial. Hence, since $h_\kappa$ is invariant under the symmetric group, it follows

$$\begin{aligned}
c_h' \int_{S^d} \left[S_{ne_1}(x)\right]^2 h_\kappa^2(x)\, d\omega &= (d+1) c_h' \int_{S^d} \left[\frac{C_n^{(\lambda,\kappa)}(x_1)}{k_n^{(\lambda,\kappa)}}\right]^2 h_\kappa^2(x)\, d\omega \\
&= \frac{(d+1) h_n^{(\lambda,\kappa)}}{\left[k_n^{(\lambda,\kappa)}\right]^2},
\end{aligned}$$

as in the proof of Corollary 4.4. If $n = 2m$, then the integrals in the second sum can be evaluated as in Lemma 4.5, so that we get

$$c_h' \int_{S^d} \left[S_{ne_1}(x)\right]^2 h_\kappa^2(x)\, d\omega$$

$$= (d+1)c'_h \int_{S^d} \left[ \frac{C_n^{(\lambda,\kappa)}(x_1)}{k_n^{(\lambda,\kappa)}} \right]^2 h_\kappa^2(x)\, d\omega$$

$$+ d(d+1)c'_h \int_{S^d} \left[ \frac{C_n^{(\lambda,\kappa)}(x_1) C_n^{(\lambda,\kappa)}(x_2)}{k_n^{(\lambda,\kappa)} k_n^{(\lambda,\kappa)}} \right]^2 h_\kappa^2(x)\, d\omega$$

$$= (d+1) \frac{h_n^{(\lambda,\kappa)}}{\left[k_n^{(\lambda,\kappa)}\right]^2} \left( 1 + d(-1)^m \frac{(\kappa + \frac{1}{2})_m}{(\lambda + \frac{1}{2})_m} \right).$$

Using the formulae $k_n^{(\lambda,\kappa)}$ in (2.7) and $h_n^{(\lambda,\kappa)}$ in (4.3) completes the proof. □

In [1], some invariant polynomials of lower degrees with the least $L^p(S^d; d\omega)$ norm on the sphere are studied. In particular, for the $L^2$ norm, it is computed there that

$$\inf_{Q \in \Pi_3^m} \|x_1^4 + \cdots + x_m^4 - Q(x)\|_2^2 = \frac{24(m-1)}{(m+2)^2(m+4)(m+6)}.$$

This is our general result with $\kappa = 0$, $n = 4$ and $m = d+1$.

The Proposition 4.6 gives the norm of the symmetric monomial polynomial $S_{ne_1}$. We do not have a compact formula for the norm of the symmetric monomial orthogonal polynomials in general.

### *4.2. Norm of monomial polynomials on the ball*

For $\alpha \in \mathbb{N}_0^d$, the polynomial $R_\alpha^B(x)$ is related to the best approximation to $x^\alpha$. Let $w_\kappa^B$ denote the normalization constant of the weight function $W_\kappa^B$ in (1.2). Define

$$\|f\|_{2,B} = \left( w_\kappa^B \int_{B^d} |f(x)|^2 W_\kappa^B(x)\, dx \right)^{1/2}, \quad w_\kappa^B = \frac{\Gamma(|\kappa| + (d+1)/2)}{\prod_{i=1}^{d+1} \Gamma(\kappa_i + 1/2)}.$$

As it is shown in the previous section, for $\alpha \in \mathbb{N}_0^d$, the monomial orthogonal polynomials $R_\alpha^B$ is related to the $h$-harmonic polynomial $R_{(\alpha,0)}$ by the formula $R_\alpha^B(x) = R_{(\alpha,0)}(x, x_{d+1})$, $(x, x_{d+1}) \in S^d$. Using the formula

$$\int_{S^d} f(y)\, d\omega = \int_{B^d} \left[ f(x, \sqrt{1-\|x\|^2}) + f(x, -\sqrt{1-\|x\|^2}) \right] \frac{dx}{\sqrt{1-\|x\|^2}},$$

the norm of $R_\alpha$ follows from that of $R_{(\alpha,0)}$ right away.

**Theorem 4.7.** *The polynomial $R_\alpha^B$ has the smallest $\|f\|_{2,B}$ norm among all polynomials of the form $x^\alpha - P(x)$, $P \in \Pi_{n-1}^d$. Furthermore, for $\alpha \in \mathbb{N}_0^d$,*

$$\|R_\alpha^B\|_{2,B}^2 = \|R_{(\alpha,0)}\|_2 = \frac{2\rho \prod_{i=1}^d \left(\kappa_i + \frac{1}{2}\right)_{\alpha_i}}{(\rho)_{|\alpha|}} \int_0^1 \prod_{i=1}^d \frac{C_{\alpha_i}^{(\frac{1}{2},\kappa_i)}(t)}{k_{\alpha_i}^{(\frac{1}{2},\kappa_i)}}\, t^{|\alpha|+2\rho-1}\, dt.$$

For the classical weight function $W_\mu(x) = (1 - \|x\|^2)^{\mu-1/2}$, the norm of $R_\alpha$ can be expressed as the integral of the product Legendre polynomials $P_n(t) = C_n^{1/2}(t)$. Equivalently, as the best approximation in the $L^2$ norm, it gives the following:

**Corollary 4.8.** *Let $\rho = \mu + (d-1)/2 > 0$ and $n = |\alpha|$ for $\alpha \in \mathbb{N}_0^d$. For the classical weight function $W_\mu(x) = (1 - \|x\|^2)^{\mu-1/2}$ on $B^d$,*

$$\min_{Q \in \Pi_{n-1}^d} \|x^\alpha - Q(x)\|_{2,B}^2 = \frac{\rho\alpha!}{2^{n-1}(\rho)_n} \int_0^1 \prod_{i=1}^d P_{\alpha_i}(t) t^{n+2\rho-1}\, dt.$$

**Proof.** Set $\kappa_i = 0$ for $1 \leqslant i \leqslant d$ and $\mu = \kappa_{d+1}$ in the formula of Theorem 4.7. The stated formula follows from $(1)_{2n} = 2^{2n}(1/2)_n(1)_n$, $n! = (1)_n$, and the fact that $C_m^{(1/2,0)}(t) = C_m^{1/2}(t) = P_m(t)$. □

In particular, for $d = 2$, the product involves only two Legendre polynomials. Since $P_n(t) = P_n^{(0,0)}(t)$, the integral for $d = 2$ can be written as a terminating ${}_4F_3$ series using the formula in (4.4) and (4.5). For the unit weight function on $B^2$ (that is, $W_{1/2}^B(x) = 1$), another formula of $R_\alpha$ is given in [3], writing it in terms of the basis $\{U_n(\cos(k\pi/(n+1))x_1 + \sin(k\pi/(n+1))x_2) : 0 \leqslant k \leqslant n\}$, where $U_n$ denotes the Chebyshev polynomial of the second kind, and the norm of $R_\alpha$, $|\alpha| = n$, is given as follows in [3]

$$\begin{aligned} &\min_{P \in \Pi_{n-1}^d} \int_{B^2} |x^\alpha - P(x)|^2\, dx \\ &\quad = \frac{n+1}{2^{2n+3}} \int_0^{2\pi} \left( \int_{-1}^1 (\sin\theta - is\cos\theta)^{\alpha_1} (\cos\theta + is\sin\theta)^{\alpha_2}\, ds \right)^2 d\theta, \end{aligned}$$

in which $\alpha = (\alpha_1, \alpha_2)$ and $i = \sqrt{-1}$. This formula is quite different from the one contained in Corollary 4.8. In fact, it is not all clear how to derive one from the other.

Setting $\alpha = ne_i$ in Theorem 4.7 and using Corollary 4.4, it follows that $\|R_{ne_i}^B\|_{2,B}^2 = h_n^{(\lambda_1,\kappa_1)} \Big/ \left[k_n^{(\lambda_1,\kappa_1)}\right]^2$. Following the proof of Proposition 4.6 we can also compute the norm of $S_{ne_1}^B$ with respect to $W_{\kappa,\mu}^B$. The result is essentially the same as in Proposition 4.6 with $d+1$ replaced by $d$, $\kappa_1 = \cdots = \kappa_d = \kappa$, $\mu = \kappa_{d+1}$ and $\lambda = \mu + (d-1)\kappa + (d-1)/2$.

### *4.3. Norm of monomial polynomials on the simplex*

In the case of simplex, the polynomials $R_\alpha^T$ is the orthogonal projection of $X^\alpha = x_1^{\alpha_1} \cdots x_d^{\alpha_d}(1 - |x|)^{\alpha_{d+1}}$. Let $w_\kappa^T$ denote the normalization constant of $W_\kappa^T$. Define

$$\|f\|_{2,T} = \left( w_\kappa^T \int_{T^d} |f(x)|^2 W_\kappa^T(x)\, dx \right)^{1/2}, \quad w_\kappa^T = \frac{\Gamma(|\kappa| + (d+1)/2)}{\prod_{i=1}^{d+1} \Gamma(\kappa_i + 1/2)}.$$

Let $F(x) = f(x_1^2, \ldots, x_{d+1}^2)$. Then the norm is related to the norm on $S^d$ via

$$c_h' \int_{S^d} f(x_1^2, \ldots, x_{d+1}^2) h_\kappa^2(x)\, d\omega = w_\kappa^T \int_{T^d} f(x_1, \ldots, x_d, 1 - |x|) W_\kappa^T(x)\, dx.$$

Since $R_\alpha^T(x_1^2, \ldots, x_{d+1}^2) = R_{2\alpha}(x_1, \ldots, x_{d+1})$, the norm of $R_\alpha^T$ can be derived from the norm of $R_{2\alpha}$. We use (2.5) to write $C_{2\beta_i}^{(1/2,\kappa_i)}(t) = P_{\beta_i}^{(0,\kappa_i-1/2)}(2t^2-1)$ and change variable $t^2 \mapsto r$ in the integral in Theorem 4.1 to get the following:

**Theorem 4.9.** *Let $\beta \in \mathbb{N}_0^{d+1}$ and $\rho = |\kappa| + (d-1)/2$. The polynomial $R_\beta^T$ has the smallest $\|\cdot\|_{2,T}$ norm among all polynomials of the form $X^\beta - P$, $P \in \Pi_{|\beta|-1}^d$, and the norm is given by*

$$\begin{aligned} w_\kappa^T &\int_{T^d} |R_\beta(x)|^2 W_\kappa^T(x)\, dx \\ &= \frac{\rho\left(\kappa + \frac{1}{2}\right)_{2\beta}}{(\rho)_{2|\beta|}} \sum_\gamma \frac{(-\beta)_\gamma \left(-\beta - \kappa + \frac{1}{2}\right)_\gamma}{\left(-2\beta - \kappa + \frac{1}{2}\right)_\gamma \gamma!(2|\beta| - |\gamma| + \rho)} \\ &= \frac{\rho\beta!\left(\kappa + \frac{1}{2}\right)_\beta}{(\rho)_{2|\beta|}} \int_0^1 \prod_{i=1}^{d+1} P_{\beta_i}^{(0,\kappa_i-1/2)}(2r-1) r^{|\beta|+\rho-1}\, dr. \end{aligned}$$

In particular, if $\beta_{d+1} = 0$, then the norm of $R_{(\beta,0)}(x)$ is the smallest norm among all polynomials of the form $x^\beta - P$, $P \in \Pi_{n-1}^d$.

**Corollary 4.10.** *Let $\alpha \in \mathbb{N}_0^d$ and $n = |\alpha|$. Then*

$$\begin{aligned} \inf_{Q \in \Pi_{n-1}^d} \|x^\alpha - Q(x)\|_{2,T}^2 = &\frac{\rho\alpha! \prod_{i=1}^d \left(\kappa_i + \frac{1}{2}\right)_{\alpha_i}}{(\rho)_{2|\alpha|}} \\ &\times \int_0^1 \prod_{i=1}^d P_{\alpha_i}^{(0,\kappa_i-1/2)}(2r-1) r^{|\alpha|+\rho-1}\, dr. \end{aligned}$$

The case $\kappa_i = \frac{1}{2}$ for $1 \leqslant i \leqslant d+1$ corresponds to the unit weight function $W_\kappa^T(x) = 1$, for which the norm is computed by an integral of the product of Legendre polynomials $P_n(t) = P_n^{(0,0)}(t)$. Indeed, setting $\kappa_i = \frac{1}{2}$ in the above theorem gives $\rho = d$ and the following:

**Corollary 4.11.** *For $\alpha \in \mathbb{N}_0^d$, $n = |\alpha|$,*

$$\min_{Q \in \Pi_{n-1}^d} \frac{1}{d!} \int_{T^d} |x^\alpha - Q(x)|^2\, dx = \frac{d\alpha!^2}{(d)_{2n}} \int_0^1 \prod_{i=1}^d P_{\alpha_i}(2r-1) r^{n+d-1}\, dr.$$

For $d = 2$, the product involves only two Jacobi polynomials, and its integral can be written using the formula in (4.4) and (4.5) in terms of a terminating ${}_4F_3$ series (setting $\sigma_i = 0$ and $a = 1$).

## 5. Expansion of $R_\alpha$ in terms of an orthonormal basis

The elements of the set $\{R_\alpha : |\alpha| = n, \alpha \in \mathbb{N}_0^{d+1}\}$ are not linearly independent, since the number of elements in the set is greater than the dimension of $\mathcal{H}_n^{d+1}(h_\kappa^2)$. It contains a basis as shown in Proposition 2.5. The basis is not orthonormal, however, since its elements are orthogonal to lower degree polynomials but not among themselves. On the other hand, an orthonormal basis for $\mathcal{H}_n^{d+1}(h_\kappa^2)$ can be given explicitly in terms of the generalized Gegenbauer polynomials $C_n^{(\lambda,\mu)}$. We first state this basis then derive the expansion of $R_\alpha$ in terms of it.

For $d \geqslant 1$, $\kappa \in \mathbb{R}^{d+1}$ and $\alpha \in \mathbb{N}_0^d$, we introduce the notation

$$\alpha^j = (\alpha_j, \ldots, \alpha_d) \quad \text{and} \quad \kappa^j = (\kappa_j, \ldots, \kappa_{d+1}), \quad 1 \leqslant j \leqslant d+1. \tag{5.1}$$

Since $\kappa^{d+1}$ consists of only the last element of $\kappa$, write $\kappa^{d+1} = \kappa_{d+1}$. These we treat as elements in $\mathbb{N}_0^{d-j+1}$ and $\mathbb{R}^{d-j+2}$, respectively, so that the quantities $|\alpha^j|$ and $|\kappa^j|$ are defined as before. Note $|\alpha^{d+1}| = 0$. We also introduce the notation

$$a_j := a_j(\alpha, \kappa) = |\alpha^{j+1}| + |\kappa^{j+1}| + \frac{d-j}{2}, \quad 1 \leqslant j \leqslant d. \tag{5.2}$$

Note that for $\alpha \in \mathbb{N}_0^d$ and $\kappa \in \mathbb{R}_+^{d+1}$, $a_d = |\kappa^{d+1}| = \kappa_{d+1}$. Finally, for $x \in \mathbb{R}^{d+1}$, let $r = \|x\|$ and define $r_j = (x_j^2 + \cdots + x_{d+1}^2)^{1/2}$ for $1 \leqslant j \leqslant d+1$. Notice that $r_1 = r$.

**Proposition 5.1.** *An orthonormal basis of $\mathcal{H}_n^{d+1}(h_\kappa^2)$ is given by*

$$\widetilde{Y}_\alpha(x) = [A_{\alpha,\kappa}]^{-1} Y_\alpha(x;\kappa), \quad \widetilde{Y}'_\alpha(x) = [A'_{\alpha,\kappa}]^{-1} Y'_\alpha(x;\kappa),$$

*where $\alpha \in \mathbb{N}_0^d$ and $|\alpha| = n$,*

$$Y_\alpha(x;\kappa) = \prod_{j=1}^d r_j^{\alpha_j} C_{\alpha_j}^{(a_j,\kappa_j)}(x_j/r_j), \quad Y'_\alpha(x;\kappa) = x_{d+1} Y_{\alpha - e_d}(x; \kappa + e_{d+1}),$$

*in which $A'_{\alpha,\kappa} = ((\kappa_{d+1} + 1/2)/(|\kappa| + (d+1)/2))^{1/2} A_{\alpha - e_d, \kappa + e_{d+1}}$ and*

$$\left[A_{\alpha,\kappa}\right]^2 = \frac{1}{\left(|\kappa| + \frac{d+1}{2}\right)_n} \prod_{j=1}^d (a_j + \kappa_j)_{\alpha_j} C_{\alpha_j}^{(a_j,\kappa_j)}(1).$$

The formulae given above are a reformulation of the basis given in [16] (also [7, p. 198]), where they are given in spherical coordinates which corresponds to $x_j/r_j = \cos\theta_{d+1-j}$, $1 \leqslant j \leqslant d$. The formulae there are given in terms of the normalized generalized Gegenbauer polynomials $\widetilde{C}_n^{(\lambda,\mu)}(t) = (1/h_n) C_n^{(\lambda,\mu)}(t)$. The normalization constant $h_n$ is given by $h_n^2 =$

$C_n^{(\lambda,\mu)}(1)(\lambda+\mu)/(n+\lambda+\mu)$ (cf. [7, p. 27]), which is used to rewrite the formulae in [16] to the above form. The fact $a_j(\kappa+e_{d+1},\alpha-e_d)=a_j(\kappa,\alpha)$, $1\leqslant j\leqslant d-1$, and $a_d(\kappa+e_{d+1},\alpha-e_d)=a_d(\kappa,\alpha)+1$ is useful for writing down $A'_{\alpha,\kappa}$.

Since $R_\alpha\in\mathcal{H}_n^{d+1}(h_\kappa^2)$, it can be expanded in terms of the orthonormal basis of $Y_\alpha$ and $Y'_\alpha$. Below we give this expansion explicitly. To do so, we need the following formula:

**Proposition 5.2.** *Let $r_j^2=b_j^2+\cdots+b_{d+1}^2$ and let $\rho=|\kappa|+(d-1)/2$. Then*

$$\sum_{|\alpha|=n} b^\alpha \widetilde{R}_\alpha(x)=\sum_{|\nu|=n}\frac{(\rho)_n}{\prod_{j=1}^d(\kappa_j+a_j)_{\nu_j}}Y_\nu(x;\kappa)\prod_{j=1}^d r_j^{\nu_j}\frac{C_{\nu_j}^{(a_j,\kappa_j)}(b_j/r_j)}{C_{\nu_j}^{(a_j,\kappa_j)}(1)}$$
$$+x_{d+1}b_{d+1}\frac{\kappa_d+\kappa_{d+1}}{\kappa_{d+1}+\frac12}\sum_{|\nu|=n}\frac{(\rho)_n}{\prod_{j=1}^d(\kappa_j+a_j)_{\nu_j}}Y_{\widetilde{\nu}}(x,\widetilde{\kappa})$$
$$\times\prod_{j=1}^d r_j^{\widetilde{\nu}_j}\frac{C_{\widetilde{\nu}_j}^{(\widetilde{a}_j,\kappa_j)}(b_j/r_j)}{C_{\widetilde{\nu}_j}^{(\widetilde{a}_j,\kappa_j)}(1)},$$

*where $\widetilde{\nu}=\nu-e_d$, $a_j=a_j(\nu,\kappa)$ and $\widetilde{a}_j=a_j(\nu-e_d,\kappa+e_{d+1})$.*

**Proof.** By the definition of the reproducing kernel, we can write

$$P_n(h_\kappa^2;x,y)=\sum_{|\nu|=n}\left(\widetilde{Y}_\nu(x;\kappa)\widetilde{Y}_\nu(y;\kappa)+\widetilde{Y}'_\nu(x;\kappa)\widetilde{Y}'_\nu(y;\kappa)\right).$$

Hence, the second part of Proposition 2.2 shows that

$$\sum_{|\alpha|=n} b^\alpha\widetilde{R}_\alpha(x)=\frac{\rho}{n+\rho}\sum_{|\nu|=n}\left(\widetilde{Y}_\nu(x;\kappa)\widetilde{Y}_\nu(y;\kappa)+\widetilde{Y}'_\nu(x;\kappa)\widetilde{Y}'_\nu(y;\kappa)\right).$$

Hence, the stated results follows from the explicit formula of $Y_\nu(x;\kappa)$ and $Y'_\nu(x;\kappa)$, $(\rho+1)_n=(\rho)_n(n+\rho)/\rho$, and checking the constants.

This proposition shows that to expand $R_\alpha$ in terms of $Y_\alpha(x;\kappa)$ we essentially have to work out the expansion of $\prod_{j=1}^d r_j^{\nu_j}C_{\nu_j}^{(a_j,\kappa_j)}(b_j/r_j)$ in power of $b$. Furthermore, the relation in (2.5) shows that

$$C_{2n+1}^{(\lambda,\mu)}(x)/C_{2n+1}^{(\lambda,\mu)}(1)=xC_{2n}^{(\lambda,\mu+1)}(x)/C_{2n}^{(\lambda,\mu+1)}(1).$$

Hence, introducing the notation $\varepsilon(\alpha)=\alpha-2[\alpha/2]$, or equivalently,

$$\varepsilon(\alpha)=(\varepsilon_1(\alpha),\ldots,\varepsilon_{d+1}(\alpha))\quad\text{with}\quad\varepsilon_i(\alpha)=\begin{cases}0 & \text{if } \alpha_i \text{ is even},\\ 1 & \text{if } \alpha_i \text{ is odd},\end{cases}$$

we can write for $\nu\in\mathbb{N}_0^d$ and $b\in\mathbb{R}^{d+1}$,

$$\prod_{j=1}^d r_j^{\nu_j}\frac{C_{\nu_j}^{(a_j,\kappa_j)}(b_j/r_j)}{C_{\nu_j}^{(a_j,\kappa_j)}(1)}=b^{\varepsilon(\nu^*)}\prod_{j=1}^d r_j^{2[\nu_j/2]}\frac{C_{2[\nu_j/2]}^{(a_j,\kappa_j+\varepsilon_j(\nu))}(b_j/r_j)}{C_{2[\nu_j/2]}^{(a_j,\kappa_j+\varepsilon_j(\nu))}(1)}$$

$$= b^{\varepsilon(\nu^*)} \prod_{j=1}^{d} r_j^{2[\nu_j/2]} \frac{P_{[\nu_j/2]}^{(a_j-\frac{1}{2},\kappa_j+\varepsilon_j(\nu)-\frac{1}{2})}(2b_j^2/r_j^2-1)}{P_{[\nu_j/2]}^{(a_j-\frac{1}{2},\kappa_j+\varepsilon_j(\nu)-\frac{1}{2})}(1)}, \tag{5.3}$$

where $\nu^* = (\nu, 0) \in \mathbb{N}_0^{d+1}$ and $r_j^2 = b_j^2 + \cdots + b_{d+1}^2$. Consequently, the problem reduces to find the power expansion of the product of the Jacobi polynomials.

The expansion can be derived using the Hahn polynomials of several variables studied by Karlin and McGregor [13]. For one variable, the Hahn polynomial $Q(x; a, b, N)$ is defined using the ${}_3F_2$ series by

$$Q_n(x; a, b, N) := {}_3\widetilde{F}_2\left( \begin{matrix} -n, n+a+b+1, -x \\ a+1, -N \end{matrix}; 1 \right), \quad n = 0, 1, \ldots, N, \tag{5.4}$$

where ${}_3\widetilde{F}_2$ is defined as the usual ${}_3F_2$ with the summation terminating at $N$. These polynomials are the discrete orthogonal polynomials defined on the set $\{0, 1, \ldots, N\}$, which are orthogonal with respect to the binomial distribution, i.e.,

$$\sum_{x=0}^{N} \frac{(a+1)_x (b+1)_{N-x}}{x!(N-x)!} Q_n(x; a, b, N) Q_m(x; a, b, N)$$
$$= \frac{(-1)^n n!(b+1)_n (n+a+b+1)_{N+1}}{N!(2n+a+b+1)(-N)_n (a+1)_n} \delta_{n,m}, \quad n, m \leqslant N.$$

A generating function for the Hahn polynomials of one variable is [12]

$$(1+t)^N \frac{P_j^{(a,b)}(\frac{1-t}{1+t})}{P_j^{(a,b)}(1)} = \sum_{n=0}^{N} \binom{N}{n} Q_j(n; a, b, N) t^n. \tag{5.5}$$

For several variables we denote the Hahn polynomials by $\phi_\nu(\alpha; \sigma, N)$. These are discrete orthogonal polynomials indexed by $\nu \in \mathbb{N}_0^d$, $|\nu| \leqslant N$ which are defined on the set $\{\alpha \in \mathbb{N}_0^{d+1} : |\alpha| = N\}$ and are orthogonal with respect to the binomial distribution given by the parameter $\sigma = (\sigma_1, \ldots, \sigma_{d+1})$. They are defined by the following generating function,

**Definition 5.3.** Suppose $\sigma \in \mathbb{R}^{d+1}$ with $\sigma_i > -1$, and $N \in \mathbb{N}$. For $\nu \in \mathbb{N}_0^d$, $|\nu| \leqslant N$ define the Hahn polynomials $\phi_\nu(\alpha; \sigma, N)$ at $\alpha \in \mathbb{N}_0^{d+1}$, $|\alpha| = N$ by

$$|y|^{N-|\nu|} \prod_{j=1}^{d} |y^j|^{\nu_j} \frac{P_{\nu_j}^{(b_j,\sigma_j)}(2y_j/|y^j|-1)}{P_{\nu_j}^{(b_j,\sigma_j)}(1)} = \sum_{|\alpha|=N} \frac{N!}{\alpha!} \phi_\nu(\alpha; \sigma, N) y^\alpha, \quad y \in \mathbb{R}^{d+1},$$

where $|y^j| = y_j + \cdots + y_{d+1}$ and $b_j = a_j(2\nu, \sigma + \frac{1}{2}) - \frac{1}{2}$ with $a_j$ as in (5.2).

Setting $d = 1$, $y_1 = t$, $y_2 = 1$ in the above, the left-hand side becomes

$$(1+t)^{N-\nu_1}(1+t)^{\nu_1} \frac{P_{\nu_1}^{(\sigma_2,\sigma_1)}(2t/(t+1)-1)}{P_{\nu_1}^{(\sigma_2,\sigma_1)}(1)} = (1+t)^N \frac{(-1)^{\nu_1} P_{\nu_j}^{(\sigma_2,\sigma_1)}\left(\frac{1-t}{1+t}\right)}{P_{\nu_1}^{(\sigma_2,\sigma_1)}(1)}$$

and the right-hand side becomes

$$\sum_{\alpha_1=0}^{N} \frac{N!}{\alpha_1!(N-\alpha_1)!} \phi_v(\alpha_1, N-\alpha_1; \sigma_1, \sigma_2, N) x_1^{\alpha_1} x_2^{N-\alpha_1}$$
$$= \sum_{\alpha_1=0}^{N} \binom{N}{\alpha_1} \phi_v(\alpha_1, N-\alpha_1; \sigma_1, \sigma_2, N) t^{\alpha_1}.$$

Hence, $\phi_{v_1}(\alpha_1, N-\alpha_1; \sigma_1, \sigma_2, N) = (-1)^{v_1} Q_{v_1}(\alpha_1; \sigma_2, \sigma_1, N)$.

Let us indicate how our definition agrees with that given in [13]. There the generating function is denoted by $G_{r,N}\left(\begin{smallmatrix}\bar{w}\\ \bar{\alpha}\end{smallmatrix}\middle|\,\bar{v}\right)$, which is defined by an inductive formula (see [13, (5.7), p. 278] and the first equation on p. 279). We make the following substitutions: $r = d+1$, $\bar{\alpha} = (\sigma_{d+1}, \sigma_d, \ldots, \sigma_1)$, $\bar{w} = (y_{d+1}, y_d, \ldots, y_1)$, $\bar{v} = (v_d, v_{d-1}, \ldots, v_1)$, and work out the generating function explicitly to obtain the form presented in Definition 5.3. Although, we will not use the explicit formulae or the orthogonal relation of $\phi_v(\alpha; \sigma, N)$, we state them below for completeness and for future reference. Both are stated in [13] by inductive formulae, from which the explicit formulae can be worked out using the aforementioned substitutions. Further simplification leads to the formula for $\phi_v(\alpha; \sigma, N)$ presented below.

**Proposition 5.4.** *For $\alpha \in \mathbb{N}_0^{d+1}$, $|\alpha| = N$ and $v \in \mathbb{N}_0^d$, $|v| \leqslant N$,*

$$\phi_v(\alpha; \sigma, N) = \frac{(-1)^{|v|}}{(-|\alpha|)_{|v|}} \prod_{j=1}^{d} \frac{(\sigma_j+1)_{v_j}}{(a_j+1)_{v_j}} (-|\alpha^j| + |v^{j+1}|)_{v_j}$$
$$\times Q_{v_j}(\alpha_j; \sigma_j, a_j, |\alpha^j| - |v^{j+1}|).$$

The proof that the $\phi_v(\alpha; \sigma, N)$ are orthogonal with respect to the binomial distribution is given in [13] and the constant $B_v$ below is given by inductive formulae (5.13), (5.14), (5.18) in [13]. The verification (using the substitution that we mentioned earlier) is straightforward.

**Proposition 5.5.** *For $v, \mu \in \mathbb{N}_0^d$ with $|v|, |\mu| \leqslant N$,*

$$\sum_{|\alpha|=N} \frac{(\sigma+1)_\alpha}{\alpha!} \phi_v(\alpha; \sigma, N) \phi_\mu(\alpha; \sigma, N) = B_v \delta_{v,\mu},$$

*where $B_v$ is given by*

$$B_v := \frac{(-1)^{|v|}(|\sigma|+d+1)_{N+|v|}}{(-N)_{|v|}\, N!\, (|\sigma|+d+1)_{2|v|}} \prod_{j=1}^{d} \frac{(\sigma_j+b_j+1)_{2v_j}(\sigma_j+1)_{v_j} v_j!}{(\sigma_j+b_j+1)_{v_j}(b_j+1)_{v_j}}.$$

For other properties of these polynomials, such as recurrence relations, see [13].

Using Proposition 5.4 and 5.3, we can now derive the expansion of $R_\alpha$ in terms of the orthonormal basis $Y_v$. Recall that $\varepsilon(\alpha) = \alpha - 2[\alpha/2]$.

**Proposition 5.6.** *For $\nu \in \mathbb{N}_0^d$ let $\rho = |\kappa| + (d-1)/2$ and let $\nu^* = (\nu, 0) \in \mathbb{N}_0^{d+1}$. Let $\alpha \in \mathbb{N}_0^{d+1}$. If $\alpha_{d+1}$ is an even integer, then*

$$R_\alpha(x) = \left(\kappa + \frac{\mathbf{1}}{2}\right)_{[\frac{\alpha+\mathbf{1}}{2}]} \sum_{\substack{|\nu|=|\alpha| \\ \varepsilon(\nu^*)=\varepsilon(\alpha)}} \frac{(|[\nu/2]|)!}{\prod_{i=1}^d (\kappa_i + a_i)_{\nu_i}} \times \phi_{[\frac{\nu}{2}]}\left(\left[\frac{\alpha}{2}\right], \kappa - \frac{\mathbf{1}}{2} + \varepsilon(\nu^*), \left|\left[\frac{\nu}{2}\right]\right|\right) Y_\nu(x;\kappa)$$

*and if $\alpha_{d+1}$ is an odd integer, then*

$$R_\alpha(x) = \left(\kappa + \frac{\mathbf{1}}{2}\right)_{[\frac{\alpha+\mathbf{1}}{2}]} \frac{\kappa_d + \kappa_{d+1}}{\kappa_{d+1} + \frac{1}{2}} \sum_{\substack{|\nu|=|\alpha| \\ \varepsilon(\widetilde{\nu}^*)=\varepsilon(\widetilde{\alpha})}} \frac{(|[\widetilde{\nu}/2]|)!}{\prod_{i=1}^d (\kappa_i + a_i)_{\nu_i}} \times \phi_{[\frac{\widetilde{\nu}}{2}]}\left(\left[\frac{\alpha}{2}\right], \widetilde{\kappa} - \frac{\mathbf{1}}{2} + \varepsilon(\widetilde{\nu}^*), \left|\left[\frac{\widetilde{\nu}}{2}\right]\right|\right) Y_\nu(x;\kappa),$$

*where $\widetilde{\nu} = \nu - e_d$, $\widetilde{\kappa} = \kappa + e_{d+1}$ and $\widetilde{\alpha} = \alpha + e_{d+1}$.*

**Proof.** Using (5.3) and the Definition 5.3 we can expand the right-hand side of the formula in Proposition 5.2 in powers of $b$. There are two terms, the first one contains only even powers of $b_{d+1}$ and the second contains only odd powers. Hence, we need to consider the two cases separately. For example, setting $\sigma_i = \kappa_i - \frac{1}{2}$ and $y_j = b_j^2$ so that $|y^j| = r_j^2$, the Definition 5.3 and (5.3) gives

$$\prod_{j=1}^d r_j^{\nu_j} \frac{C_{\nu_j}^{(a_j,\kappa_j)}(b_j/r_j)}{C_{\nu_j}^{(a_j,\kappa_j)}(1)} = \sum_{|\beta|=|[\nu/2]|} \frac{(|[\nu/2]|)!}{\beta!} \times \phi_{[\frac{\nu}{2}]}\left(\beta, \kappa - \frac{\mathbf{1}}{2} + \varepsilon(\nu^*), \left|\left[\frac{\nu}{2}\right]\right|\right) b^{2\beta+\varepsilon(\nu^*)},$$

which gives the expansion of the first term in the right-hand side of the formula in Proposition 5.2. That is, for $\alpha_{d+1}$ being even,

$$\sum_{\substack{|\alpha|=n \\ \alpha_{d+1}=\text{even}}} b^\alpha \widetilde{R}_\alpha(x) = \sum_{|\nu|=n} \sum_{|\beta|=|[\nu/2]|} \frac{(\rho)_{|\nu|}}{\prod_{i=1}^d (\kappa_i + a_i)_{\nu_i}} \frac{(|[\nu/2]|)!}{\beta!} \times \phi_{[\frac{\nu}{2}]}\left(\beta, \kappa - \frac{\mathbf{1}}{2} + \varepsilon(\nu^*), \left|\left[\frac{\nu}{2}\right]\right|\right) Y_\nu(x;\kappa) b^{2\beta+\varepsilon(\nu^*)}.$$

To derive the formula of $R_\alpha$, we set $2\beta + \varepsilon(\nu^*) = \alpha = 2[\alpha/2] + \varepsilon(\alpha)$. This gives $\beta = [\alpha/2]$ and $\varepsilon(\nu^*) = \varepsilon(\alpha)$, so that

$$\widetilde{R}_\alpha(x) = \sum_{\substack{|\nu|=|\alpha| \\ \varepsilon(\nu^*)=\varepsilon(\alpha)}} \frac{(\rho)_{|\nu|}}{\prod_{i=1}^d (\kappa_i + a_i)_{\nu_i}} \frac{(|[\nu/2]|)!}{([\alpha/2])!} \times \phi_{[\frac{\nu}{2}]}\left(\left[\frac{\alpha}{2}\right], \kappa - \frac{\mathbf{1}}{2} + \varepsilon(\nu^*), \left|\left[\frac{\nu}{2}\right]\right|\right) Y_\nu(x;\kappa).$$

Then we use the relation in Proposition 2.3 to replace $\widetilde{R}_\alpha$ by $R_\alpha$. The constant is simplified by the fact that $\alpha! = 2^{|\alpha|}(\mathbf{1}/2)_\beta(\alpha/2)!$, $\beta = [\frac{\alpha+\mathbf{1}}{2}]$. The case of $\alpha_{d+1}$ is odd is proved similarly. □

The expansion of $R_\alpha^B$ or $R_\alpha^T$ in terms of an explicit orthonormal basis can be derived from the above proposition. We give the result for $R_\alpha^T$ below. First we state an orthonormal basis with respect to $W_\kappa^T$.

**Definition 5.7.** For $v \in \mathbb{N}_0^d$ and $x \in \mathbb{R}^d$,

$$P_v(x) := \prod_{j=1}^{d} (1 - |\mathbf{x}_{j-1}|)^{v_j} P_{v_j}^{(a_j-1/2,\kappa_j-1/2)}\left(\frac{2x_j}{1-|\mathbf{x}_{j-1}|} - 1\right),$$

where $|\mathbf{x}_j| = x_1 + \cdots + x_j$ for $1 \leqslant j \leqslant d$ and $\mathbf{x}_0 := 0$.

The set $\{P_v : |v| = n\}$ is a basis for orthogonal polynomials of degree $n$ with respect to $W_\kappa^T$ and the elements in the set are mutually orthogonal; see, for example, [7, p. 47]. The $L^2$ norm of $P_v$ is given by

$$w_\kappa^T \int_{T^d} |P_v(x)|^2 W_\kappa^T(x)\,dx = \frac{1}{(|\kappa| + \frac{d+1}{2})_{2|v|}} \prod_{j=1}^{d} \frac{(\kappa_j + a_j)_{2v_j}(a_j + 1/2)_{v_j}(\kappa_j + 1/2)_{v_j}}{(\kappa_j + a_j)_{v_j} v_j!}.$$

Under the correspondence (2.9), the polynomial $P_v$ is related to $Y_{2v}(x,\kappa)$ in Proposition 5.1. In fact, for $x = (x_1, \ldots, x_d, x_{d+1}) \in S^d$, the relation (2.5) gives

$$Y_{2v}(x,\kappa) = \prod_{i=1}^{d} \frac{(\kappa_i + a_i)_{v_i}}{(\kappa_i + 1/2)_{v_i}} P_v(x_1^2, \ldots, x_d^2),$$

where we have used the fact that $r_j^2 = 1 - x_1^2 - \cdots - x_{j-1}^2$ if $\|x\| = 1$. Hence,

**Corollary 5.8.** *Let $\rho = |\kappa| + (d-1)/2$. For $\alpha \in \mathbb{N}_0^{d+1}$, $n = |\alpha|$,*

$$R_\alpha^T(x) = n!\left(\kappa + \frac{1}{2}\right)_\alpha \sum_{|v|=n} \prod_{i=1}^{d} \frac{(\kappa_i + a_i)_{v_i}}{(\kappa_i + a_i)_{2v_i}(\kappa_i + \frac{1}{2})_{v_i}} \phi_v\left(\alpha, \kappa - \frac{1}{2}, n\right) P_v(x).$$

**Proof.** Using the fact $R_{2\alpha}(x) = R_\alpha^T(x_1^2, \ldots, x_{d+1}^2)$, the formula comes from $R_{2\alpha}$ in the Proposition 5.6. Note that $\varepsilon(v^*) = \varepsilon(2\alpha)$ implies that $v_i$ is even for every $i$. □

Since polynomials $Y_v$ are mutually orthogonal, the expansion in Proposition 5.6 can be used to compute the inner product of $Y_v$ and $R_\alpha$. Similarly, the Corollary 5.8 can be used to compute the inner product of $R_\alpha^T$ and $P_v$.

## 6. Further results

The definition of $R_\alpha$ in Definition 2.1 makes sense for $h$-harmonics associated with other reflection groups. For background on the theory of $h$-harmonics in general, see [4,5,7] and the references therein. Although a formula of the intertwining operator $V_\kappa$ is unknown in general, it is known that $V_\kappa$ is a bounded operator in the following sense ([5]). Let $\|p\|_\infty := \sup_{B^{d+1}} |p(x)|$ for any polynomial $p$. For formal sums $f(x) = \sum_{n=0}^\infty f_n(x)$ with $f_n \in \mathcal{P}_n^{d+1}$, let $\|f\|_A := \sum_{n=0}^\infty \|f_n\|_\infty$ and let $\mathcal{A} := \{f : \|f\|_A < \infty\}$. Then for $f \in \mathcal{A}$, $|Vf(x)| \leqslant \|f\|_A$ for $x \in B^{d+1}$. This fact can be used to justify the definition of $R_\alpha$ in Definition 2.1 for other reflection groups.

We will not discuss $R_\alpha$ associated with general reflection groups any further, but merely point out that Proposition 2.2 holds in the general setting and prove one more such result which gives the expansion of $Vx^\alpha$ in terms of $R_\beta$. Recall that Proposition 2.2 shows

$$R_\alpha(x) = \sum_\gamma \frac{(-\alpha/2)_\gamma((-\alpha+\mathbf{1})/2)_\gamma}{(-|\alpha|-\rho+1)_{|\gamma|}\gamma!} \|x\|^{2|\gamma|} V_\kappa x^{\alpha-2\gamma}.$$

The following proposition states that the above expansion can be reversed.

**Proposition 6.1.** *Let* $\alpha \in \mathbb{N}_0^{d+1}$. *Then*

$$V_\kappa x^\alpha = \sum_{2\beta\leqslant\alpha} \frac{(-1)^{|\beta|}(-\alpha/2)_\beta((-\alpha+\mathbf{1})/2)_\beta}{(-|\alpha|-\rho+1)_{2|\beta|}\beta!} \times \left(2(-|\alpha|-\rho+1)_{|\beta|} - (-|\alpha|-\rho)_{|\beta|}\right) \|x\|^{2|\beta|} R_{\alpha-2\beta}(x).$$

**Proof.** We show that there exist $a_\beta$ such that $a_0 = 1$ and

$$V_\kappa x^\alpha = \sum_{2\beta\leqslant\alpha} a_\beta \|x\|^{2|\beta|} R_{\alpha-2\beta}(x),$$

the values of $a_\beta$ will be uniquely determined as the stated value. Using the formula $R_\alpha(x) = \sum_\gamma c_{\alpha,\gamma}\|x\|^{2|\gamma|}V_\kappa x^{\alpha-2\gamma}$, it follows that

$$\sum_{2\beta\leqslant\alpha} a_\beta\|x\|^{2|\beta|}R_{\alpha-2\beta}(x) = \sum_{2\beta\leqslant\alpha} a_\beta \sum_{2\gamma\leqslant\alpha-2\beta} c_{\alpha-2\beta,\gamma}\|x\|^{2|\beta|+2|\gamma|}V_\kappa x^{\alpha-2\beta-2\gamma}$$
$$= \sum_{2\gamma\leqslant\alpha} V_\kappa x^{\alpha-2\gamma}\|x\|^{2|\gamma|} \sum_{\beta\leqslant\gamma} a_\beta c_{\alpha-2\beta,\gamma-\beta}.$$

Since $(a+m)_{n-m} = (a)_n/(a)_m$, we have

$$c_{\alpha-2\beta,\gamma-\beta} = \frac{(-|\alpha|-\rho+1)_{2|\beta|}}{(-\alpha/2)_\beta((-\alpha+\mathbf{1})/2)_\beta(-|\alpha|-\rho+1)_{|\gamma|+|\beta|}(\gamma-\beta)!}$$

so that we need to show that there exist $a_\beta$ such that $a_0 = 1$ and

$$\Sigma_\gamma := \sum_{\beta\leqslant\gamma} \frac{a_\beta^*}{(-|\alpha|-\rho+1)_{|\gamma|+|\beta|}(\gamma-\beta)!\beta!} = 0,$$

$$a^*_\beta = \frac{(-|\alpha| - \rho + 1)_{2|\beta|}\beta!}{(-\alpha/2)_\beta((-\alpha + \mathbf{1})/2)_\beta} a_\beta$$

for $\gamma \neq 0$. For each $\Sigma_\gamma$, $a^*_\gamma$ has the dominating subindex among all $a^*_\beta$ in $\Sigma_\gamma$. Consequently, one can solve for $a^*_\gamma$ recursively from the equations $\Sigma_\gamma = 0$. The fact that $a^*_0 = a_0 = 1$ shows then that the solution is unique. Hence, to complete the proof, we only have to show that $a^*_\beta = (-1)^\beta(2(-|\alpha| - \rho + 1)_{|\beta|} - (-|\alpha| - \rho)_{|\beta|})$ is a solution. To do so, we need to recall the definition of another Lauricella function, the function of type $D$, defined by

$$F_D(a, \alpha; c; x) = \sum_\beta \frac{(a)_{|\beta|}(\alpha)_\beta}{(c)_{|\beta|}\beta!} x^\beta, \quad a, c \in \mathbb{R},\ \alpha \in \mathbb{N}_0^{d+1}, \quad \max_{1 \leqslant i \leqslant d+1} |x_i| < 1.$$

Then, with $a^*_\beta$ so chosen, using the fact that $(\gamma - \beta)! = (-1)^{|\beta|}\gamma!/(-\gamma)_\beta$ and $(a)_{n+m} = (a + n)_m(a)_n$, we obtain

$$\begin{aligned}
\Sigma_\gamma &= \sum_{\beta \leqslant \gamma} \frac{(-1)^\beta(2(-|\alpha| - \rho + 1)_{|\beta|} - (-|\alpha| - \rho)_{|\beta|})}{(-|\alpha| - \rho + 1)_{|\gamma|+|\beta|}(\gamma - \beta)!\beta!} \\
&= \frac{1}{\gamma!(-|\alpha| - \rho + 1)_{|\gamma|}} \sum_{\beta \leqslant \gamma} \frac{(-\gamma)_\beta(2(-|\alpha| - \rho + 1)_{|\beta|} - (-|\alpha| - \rho)_{|\beta|})}{(-|\alpha| - \rho + 1 + |\gamma|)_{|\beta|}\beta!} \\
&= \frac{1}{\gamma!(-|\alpha| - \rho + 1)_{|\gamma|}} \big[2F_D(-|\alpha| - \rho + 1, -\gamma; -|\alpha| - \rho + 1 + |\gamma|; \mathbf{1}) \\
&\qquad - F_D(-|\alpha| - \rho, -\gamma; -|\alpha| - \rho + 1 + |\gamma|; \mathbf{1})\big].
\end{aligned}$$

Using Lauricella's identity [2, p. 116] and the Chu–Vandermonde identity

$$F_D(a, \alpha; c; \mathbf{1}) = {}_2F_1(a, |\alpha|; c; 1) \quad \text{and} \quad {}_2F_1(-n, b; c; 1) = \frac{(c - b)_n}{(c)_n},$$

we can then conclude that $\Sigma_\gamma = 0$. □

For the case of $h^2_\kappa$ in (1.1), the explicit formula of the intertwining operator $V_\kappa$ gives that

$$V_\kappa x^\alpha = \frac{\left(\frac{\mathbf{1}}{2}\right)_\beta}{\left(\kappa + \frac{\mathbf{1}}{2}\right)_\beta} x^\alpha, \quad \beta = \frac{\alpha + \mathbf{1}}{2}$$

which gives the following corollary:

**Corollary 6.2.** *Let* $\alpha \in \mathbb{N}_0^{d+1}$. *For* $h^2_\kappa$ *in* (1.1) *and* $\rho = |\kappa| + (d - 1)/2$,

$$\begin{aligned}
x^\alpha = \frac{\left(\kappa + \frac{\mathbf{1}}{2}\right)_{[\frac{\alpha+\mathbf{1}}{2}]}}{\left(\frac{\mathbf{1}}{2}\right)_{[\frac{\alpha+\mathbf{1}}{2}]}} &\sum_{2\beta \leqslant \alpha} \frac{(-1)^{|\beta|}(-\alpha/2)_\beta((-\alpha + \mathbf{1})/2)_\beta}{(-|\alpha| - \rho + 1)_{2|\beta|}\beta!} \\
&\times \left(2(|-\alpha| - \rho + 1)_{|\beta|} - (-|\alpha| - \rho)_{|\beta|}\right) \|x\|^{2|\beta|} R_{\alpha - 2\beta}(x).
\end{aligned}$$

For orthogonal polynomials with respect to $W_\kappa^B$ on $B^d$ and $W_\kappa^T$ on $T^d$, we can also derive the explicit formula of the expansion of $x^\alpha$ in terms of monomial orthogonal basis. For example, we have

**Corollary 6.3.** *Let $\alpha \in \mathbb{N}_0^d$. For $W_\kappa^T$ in (1.3) and $\rho = |\kappa| + (d-1)/2$,*

$$x^\alpha = \frac{\left(\kappa + \frac{1}{2}\right)_\alpha}{\left(\frac{1}{2}\right)_\alpha} \sum_{\beta \leqslant \alpha} \frac{(-1)^{|\beta|}(-\alpha)_\beta(-\alpha + \mathbf{1}/2)_\beta}{(2|\alpha| + \rho + 1)_{2|\beta|}\beta!} \times \left(2(2|\alpha| + \rho + 1)_{|\beta|} - (2|\alpha| + \rho)_{|\beta|}\right) R^T_{\alpha-\beta}(x).$$

Let us point out that the Proposition 6.1 holds for other reflection groups, since it is a formal inverse of the definition of $R_\alpha$. Note that for other reflection groups, $V_\kappa x^\alpha$ is not a constant multiple of $x^\alpha$ in general and neither is $R_\alpha$ an orthogonal projection of $x^\alpha$. One interesting aspect of Proposition 6.1 lies in the fact that $R_\alpha$ can be computed explicitly if an orthonormal basis is known, since such a basis will give a formula for the reproducing kernel of $\mathcal{H}_n^{d+1}(h_\kappa^2)$ so that Proposition 2.2 can be used to produce a formula of $R_\alpha$. Once the formula of $R_\alpha$ is known, the formula in Proposition 6.1 gives an explicit formula of $V_\kappa x^\alpha$, which is of interest since an explicit formula of $V_\kappa$ is not known for general reflection groups. For example, in the case of dihedral group $I_{2k}$ for which

$$h_\kappa(x) = |\cos m\theta|^{\kappa_1} |\sin m\theta|^{\kappa_2}, \quad x = (\cos\theta, \sin\theta),$$

an orthonormal basis of $\mathcal{H}_n^2(h_\kappa^2)$ is known [4]. Hence, the above outline can be carried out to give an explicit formula of $V_\kappa x^\alpha$. However, the formula is complicated and it does not seem to give any indication of the explicit formula of $V_\kappa$. We shall not present them.

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## References


[1] N.N. Andreev, V.A. Yudin, Polynomials of least derivation from zero and Chebyshev-type cubature formulas, Proc. Steklov Inst. Math. 232 (2001) 45–57.

[2] P. Appell, J.K. de Fériet, Fonctions Hypergéométriques et Hypersphériques, Polynomes d'Hermite, Gauthier-Villars, Paris, 1926.

[3] B.D. Bojanov, W. Haussmann, G.P. Nikolov, Bivariate polynomials of least deviation from zero, Canad. J. Math. 53 (2001) 489–505.

[4] C.F. Dunkl, Differential–difference operators associated to reflection groups, Trans. Amer. Math. Soc. 311 (1989) 167–183.

[5] C.F. Dunkl, Integral kernels with reflection group invariance, Canad. J. Math. 43 (1991) 1213–1227.

[6] C.F. Dunkl, Symmetric functions and $B_N$-invariant spherical harmonics, J. Phys. A. Math. Gen. 35 (2002) 10391–10408.

[7] C.F. Dunkl, Yuan Xu, Orthogonal Polynomials of several Variables, Cambridge University Press, Cambridge, 2001.

[8] A. Erdélyi, W. Magnus, F. Oberhettinger, F.G. Tricomi, Higher Transcendental Functions, McGraw-Hill, New York, 1953.
[9] A. Erdélyi, W. Magnus, F. Oberhettinger, F.G. Tricomi, Tables of Integral Transforms, McGraw-Hill, New York, 1953.
[10] H. Exton, Multiple Hypergeometric Functions and Applications, Halsted, New York, 1976.
[11] E.G. Kalnins, W. Miller Jr., M.V. Tratnik, Families of orthogonal and biorthogonal polynomials on the $N$-sphere, SIAM J. Math. Anal. 22 (1991) 272–294.
[12] S. Karlin, J. McGregor, The Hahn polynomials, formulas and an application, Scripta Math. 45 (1974) 176–198.
[13] S. Karlin, J. McGregor, Linear growth models with many types and multidimensional Hahn polynomials, in: R.A. Askey (Ed.), Theory and Applications of Special Functions, Academic Press, New York, 1975, pp. 261–288.
[14] I.G. Macdonald, Symmetric Functions and Hall Polynomials, second ed., Oxford Mathematical Monographs, Clarendon Press, New York, 1995.
[15] G. Szegő, Orthogonal polynomials, fourth ed., American Mathematical Society Colloquium Publication, vol. 23, American Mathematical Society, Providence, RI, 1975.
[16] Yuan Xu. Orthogonal polynomials for a family of product weight functions on the spheres, Canad. J. Math. 49 (1997) 175–192.
[17] Yuan Xu. Orthogonal polynomials and cubature formulae on spheres and on balls, SIAM J. Math. Anal. 29 (1998) 779–793.
[18] Yuan Xu. Orthogonal polynomials and cubature formulae on spheres and on simplices, Methods Anal. Appl. 5 (1998) 169–184.
[19] Yuan Xu. Harmonic polynomials associated with reflection groups, Canad. Math. Bull. 43 (2000) 496–507.